Contents

1 Introduction

The constructions of smooth piecewise polynomial surfaces over the given tesselation of the domain create the significant part of the geometric modeling. The most often tesselation is either the division of the domain into rectangular net or it is its triangulation. Being general enough, the surfaces constructed over the triangulation are in many cases more suitable for the modeling. This is also the reason why my work is focused on them.

To model smooth surfaces is of course really much more difficult then to model smooth curves. Very big problems appear even if we want to model only C^1 continuous surfaces. It seems for instance that the space of piecewise quadratic surfaces is not very suitable for such modeling (see chapter 8).

Certain possibilities of modeling the piecewise cubic C^1 continuous surfaces are given e.g. by Honey-Comb algorithm. However, it works only on a very specially triangulated domain. For more general triangulations only to find the dimension of the corresponding space of cubic splines seems to be a hard task.

The method of finite elements is also one of the approaches to the question. All equations which the surface is supposed to satisfy, are included in the only matrix. For finding the surface with some conditions put on it we need to find a suitable solution of the matrix. At the first sight it is obvious that for larger triangulations the method is without the practical use because of its large memory and time complexity.

Very nice means for the modeling of smooth piecewise polynomial surfaces over the arbitrary triangulation are given by B-spline functions. But the significant refinement of the input triangulation is their characteristic feature. More discussion can be found in chapter 5.

There are publications describing the modeling of C^1 continuous splines over the triangulation similarly to this work. General theory with very good algebraic background can be find e.g. in [CLO98]. Further, in [Dav98], the locally independent basis is constructed. But it is done only for surfaces with degree at least 5. I am trying to solve the question already for the surfaces of fourth degree. It seem to be considerably more complex. After the introductory notions in the chapter 2 and the chapter 3, which only briefly bring into CAGD (for more details see e.g [HL93]), I devote the chapter 4 to the description of the domain over which I want to construct the surfaces. In the chapter 6 the dimension of the space of C^1 continuous spline function of at least fourth degree is computed. The I restrict my considerations only on surfaces of fourth degree and I am looking for its basis in the chapter 7. Analogously, but with much less effort, the basis for higher degree piecewise polynomials can be constructed. Several conclusion remarks are put into the chapter 9

2 Basic Notions and Definitions

2.1 Bézier Curves

Definition 2.1 *Let E^d denotes d-dimensional euclidean space. Let $t_0, t_1 \in E^1; t_0 \neq t_1$. Then for each $u \in E^1$ there exists the only number $\lambda(u) \in R$ such that*

$$u = (1 - \lambda(u))t_0 + \lambda(u)t_1 = t_0 + \lambda(u)(t_1 - t_0).$$

We call $(1 - \lambda(u))$ and $\lambda(u)$ ***barycentric coordinates*** *of the point u with respect to t_0, t_1.*

Definition 2.2 *Let $t_0, t_1 \in E^1; t_0 \neq t_1, u = (1-\lambda(u))t_0+\lambda(u)t_1$. Polynomial $B_i^n : E^1 \to R$ defined*

$$B_i^n(u) := \binom{n}{i}(1 - \lambda(u))^{n-i}\lambda(u)^i$$

is called ***i-th Bernstein polynomial of degree n****, $i = 0, \dots, n$.*

Bernstein polynomials B_i^n create, similarly to the monomial basis, a basis of all polynomial functions of degree up to n.

Definition 2.3 *The map $B : [t_0, t_1] \to E^d$ defined*

$$B(u) := \sum_{i=1}^{n} B_i^n(u) b_i,$$

*where b_i are points from E^d, represents **Bézier curve of degree n**. Points b_i are called **Bézier control points** and create the **control polygon** of Bézier curve.*

2.2 Bézier Patches

Let points $t_0, t_1, t_2 \in E^2$ are non-collinear. Then for each $u \in E^2$ there exist unique triplet of numbers $\lambda_0(u), \lambda_1(u), \lambda_2(u) \in R$, satisfying

$$u = \lambda_0(u)t_0 + \lambda_1(u)t_1 + \lambda_2(u)t_2; \qquad \lambda_0(u) + \lambda_1(u) + \lambda_2(u) = 1. \tag{1}$$

Definition 2.4 *The numbers $\lambda_0(u), \lambda_1(u), \lambda_2(u)$ defined by (1) are called **barycentric coordinates** of the point u with respect to the triangle $\triangle t_0 t_1 t_2$.*

For these numbers it holds

$$\lambda_0(u) = \frac{|\triangle u t_1 t_2|}{|\triangle t_0 t_1 t_2|}, \quad \lambda_1(u) = \frac{|\triangle t_0 u t_2|}{|\triangle t_0 t_1 t_2|}, \quad \lambda_2(u) = \frac{|\triangle t_0 t_1 u|}{|\triangle t_0 t_1 t_2|}; \tag{2}$$

where $|\triangle t_i t_j t_k|$ is the oriented area of the triangle $\triangle t_i t_j t_k$. (see [Sei92])

Sometimes instead of

$$t = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$$

we will write

$$t = (\lambda_0, \lambda_1, \lambda_2) \quad \text{w.r.t.} \triangle t_0 t_1 t_2.$$

Assume that the points t_0, t_1, t_2 are fixly given.

Definition 2.5 *Polynomial $B_{ijk}^{\triangle} : E^2 \to R$ defined*

$$B_{ijk}^{\triangle}(u) := \binom{n}{ijk} \lambda_0^i(u) \lambda_1^j(u) \lambda_2^k(u); \quad i + j + k = n$$

*is called **bivariate Bernstein polynomial of degree n**.*

Notice, that all bivariate Bernstein polynomials of degree n (i.e. $i + j + k = n$) create basis of the space of bivariate polynomials of total degree at most n.

Definition 2.6 *Let $B : E^2 \to E^d$ is defined :*

$$B(u) := \sum_{i+j+k=n} B_{ijk}^{\triangle}(u) b_{ijk}; \quad b_{ijk} \text{ are points from } E^d$$

*The restriction of the map to $\triangle t_0 t_1 t_2$ is called **triangular Bézier patch** of degree n. Points b_{ijk} are called **Bézier control points** and create a **control net**.*

2.3 Polar Forms

Definition 2.7 *A map $f : E^2 \to R$ is* ***linear*** *iff it preserves linear combinations, i. e.*

$$f(\sum \alpha_i u_i) = \sum \alpha_i f(u_i); \quad u_i \in M, \ \alpha_i \in R$$

$f : E^2 \to R$ is ***affine*** *iff it preserves affine combinations of points :*

$$f(\sum \alpha_i j_i) = \sum \alpha_i f(u_i); \quad u_i \in M, \ \alpha_i \in R, \ \sum \alpha_i = 1$$

$f : (E^2)^n \to R$ is ***multilinear (multiaffine)*** *iff f is linear (affine) in each argument when the others are fixed.*
$f : (E^2)^n \to R$ is ***symmetric*** *iff its value does not depend on the order of arguments. It means*

$$f(u_{\pi_1}, u_{\pi_2}, \dots u_{\pi_n}) = f(u_1, u_2, \dots u_n)$$

where $(\pi_1, \pi_2, \dots \pi_n)$ is a permutation of the set $(1, 2, \dots n)$.

Now we can introduce the definition of polar forms, which is based on previous notions.

Definition 2.8 *Let $F : E^2 \to E^d$ is polynomial of degree n. Then there exists unique map $f : (E^2)^n \to E^d$ which is multiaffine, symmetric and has a diagonal property, (i.e. $f(u, \dots u) = F(u)$). This map is called* ***(affine) blossom*** *(or* ***polar form****)* ***of F****.*

The blossoming grants many useful properties. We mention only few of them.

Lemma 2.9 *Let F be a Bézier patch of degree n with control points b_{ijk} for $i + j + k = n$ with respect to $\triangle t_0 t_1 t_2$ and let f is its polar form. Then*

$$b_{ijk} = f(t_0^i t_1^j t_2^k); \qquad i + j + k = n \tag{3}$$

Lemma 2.10 ***Greville Abscissae.*** *The Bézier points of the polynomial $F(u) = u$ with respect to $\triangle t_0 t_1 t_2$ are given as*

$$b_{ijk} = \frac{1}{n}(it_0 + jt_1 + kt_2) \tag{4}$$

The proofs of all these properties can be found in [Sei92].

3 Parametric Continuity

3.1 Definition and Properties

Definition 3.1 *Let $M \subset E^2, \ F : M \to E^d$. Let $u \in intM$, and $e_1, \dots, e_q$ are arbitrary vectors. Then* ***q-th directional derivative*** *of the map F with respect to the vectors $e_1, \dots, e_q$ in the point u is defined as follows:*

$$\begin{aligned} &1) \qquad D_{e_1} F(u) = \lim_{\epsilon \to 0} \frac{F(u + \epsilon e_1) - F(u)}{\epsilon} \\ &2) \qquad D_{e_1, \dots, e_i} F(u) = D_{e_i}(D_{e_1, \dots, e_{i-1}} F(u)); \ i = 1, \dots, q \end{aligned}$$

Theorem 3.2 *The directional derivative operator D is linear; i.e. if $e = \sum_{i=1}^n a_i e_i$; $a_i \in R$, $a_i \neq 0$, e, e_i are vectors, $i = 1, \dots, n$, then*

$$D_e F(u) = \sum_{i=1}^n a_i D_{e_i} F(u)$$

where u is from the interior of the domain of F.

Proof: Let $a \in R, a \neq 0,\ e_1$ is a vector. Let $e = ae_1$ then

$$\begin{aligned} D_eF(u) &= \lim_{\epsilon\to 0}\frac{F(u+\epsilon e)-F(u)}{\epsilon} = \\ &= a \lim_{\epsilon_1 = a\epsilon\to 0}\frac{F(u+\epsilon_1 e_1)-F(u)}{\epsilon_1} = \\ &= aD_{e_1}F(u) \end{aligned}$$

Now, let e_1, e_2 are vectors and let $e = e_1 + e_2$. Then

$$\begin{aligned} D_eF(u) &= \lim_{\epsilon\to 0}\frac{F(u+\epsilon e_1+\epsilon e_2)-F(u)}{\epsilon} = \\ &= \lim_{\epsilon\to 0}\frac{F(u+\epsilon e_1+\epsilon e_2)-F(u+\epsilon e_1)}{\epsilon} + \lim_{\epsilon\to 0}\frac{F(u+\epsilon e_1)-F(u)}{\epsilon} \end{aligned}$$

The second limes from the sum equals $D_{e_1}F(u)$ directly from the definition. And instead of the first limes we will compute the following one:

$$\lim_{\epsilon_1,\epsilon_2\to 0}\frac{F(u+\epsilon_1 e_1+\epsilon_2 e_2)-F(u+\epsilon_1 e_1)}{\epsilon_2} \tag{5}$$

If this limes exists, then also the limes

$$\lim_{\epsilon\to 0}\frac{F(u+\epsilon e_1+\epsilon e_2)-F(u+\epsilon e_1)}{\epsilon} \tag{6}$$

exists and equals to (5), because (6) is only the special case of (5). Because F is a polynomial, the limes (5) surely exists and can be computed as follows:

$$\begin{aligned} (5) &= \lim_{\epsilon_2\to 0}\Big(\lim_{\epsilon_1\to 0}\frac{F(u+\epsilon_1 e_1+\epsilon_2 e_2)-F(u+\epsilon_1 e_1)}{\epsilon_2}\Big) = \\ &= \lim_{\epsilon_2\to 0}\frac{F(u+\epsilon_2 e_2)-F(u)}{\epsilon_2} = D_{e_2}F(u). \end{aligned}$$

Therefore also for $e = e_1 + e_2$ holds

$$D_eF(u) = D_{e_1}F(u) + D_{e_2}F(u).$$

■

Note 3.3 *Let polynomial F is defined in $u_0 \in E^2$. Thanks to Theorem 3.2 we can compute the directional derivative $D_eF(u_0)$ even if F is not defined for any $u = u_0 + ke,\ k \neq 0$. It is enough to know two derivatives $D_{e_1}F(u_0)$, $D_{e_2}F(u_0)$ for independent vectors e_1, e_2 to be able to find $D_eF(u_0)$ for an arbitrary vector e. And thanks to that fact we can present the following definition.*

Definition 3.4 *Let $M_F, M_G \subset E^2$. Let $F : M_F \to E^d$ and $G : M_G \to E^d$ are bivariate polynomials. Let $\tilde{F}, \tilde{G} : E^2 \to E^d$ are such polynomials, that $\tilde{F}|_{M_F} = F$ and $\tilde{G}|_{M_G} = G$. Then F, G are* ***parametric continuous of the order q*** *in $u \in M_F \cap M_G$ if all respective directional derivatives of $\tilde{F}, \tilde{G}$ in u up to order q are equal, i.e.*

$$D_{e_1,\ldots,e_i}\tilde{F}(u) = D_{e_1,\ldots,e_i}\tilde{G}(u)$$

where $e_i \in R^2$ are arbitrary and $i = 1, \ldots, q$.

As we will see, the theory of polar forms can be applied on parametric continuity of surfaces.

Theorem 3.5 *(C^q* ***conditions.****) Polynomials $F, G : E^2 \to E^d$ of degree n are C^q continuous at the point $u \in E^2$ iff*

$$f(u^{n-q}u_1 \ldots u_q) = g(u^{n-q}u_1 \ldots u_q); \quad u_1, \ldots u_q \in E^2$$

Note 3.6 *We will use a simpler and clearer multiplicative notation: Instead of* $f(u_1, u_2, \dots u_n)$ *we are going to write* $f(u_1 u_2 \dots u_n)$*. By the entry* $f(u^{n-k} u_1 u_2 \dots u_k)$ *we want to say that argument* u *occurs* $(n-k)$ *times there.*

For the further considerations we need to describe two following situations:

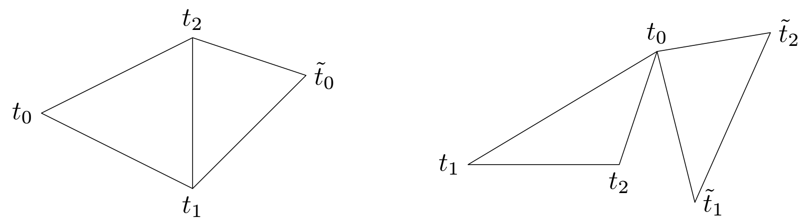

Figure 1: Two configurations of triangles.

Situation 1. Let t_0, t_1, t_2 form a proper triangle and $\tilde{t}_0$ lies outside the $\triangle t_0 t_1 t_2$ so that $\lambda_0, \lambda_1, \lambda_2$ are barycentric coordinates of $\tilde{t}_0$ with respect to $\triangle t_0 t_1 t_2$ (i.e. $\tilde{t}_0 = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$) and $\lambda_0 < 0$ (see Figure 1, left). Let Bézier patches F, G map this point configuration in the following way.

$$\begin{aligned} F &: \triangle = \triangle t_0 t_1 t_2 \to E^d; \quad F(u) = \sum B_{ijk}^{\triangle}(u) b_{ijk} \\ G &: \tilde{\triangle} = \triangle \tilde{t}_0 t_1 t_2 \to E^d; \quad G(u) = \sum B_{ijk}^{\tilde{\triangle}}(u) \tilde{b}_{ijk} \end{aligned}$$

It means that b_{ijk} are control points of the patch F w.r.t. $\triangle t_0 t_1 t_2$ and $\tilde{b}_{ijk}$ are control points of the patch G w.r.t. $\triangle \tilde{t}_0 t_1 t_2$.

Situation 2. Let t_0, t_1, t_2 and $t_0, \tilde{t}_1, \tilde{t}_2$ form two triangles with common vertex t_0 (see Figure 1, right). $\lambda_{i0}, \lambda_{i1}, \lambda_{i2}$ are barycentric coordinates of $\tilde{t}_i$; $i = 1, 2$ with respect to $\triangle t_0 t_1 t_2$, i.e. $\tilde{t}_i = \lambda_{i0} t_0 + \lambda_{i1} t_1 + \lambda_{i2} t_2$. Bézier patches F, G are now defined as follows

$$\begin{aligned} F &: \triangle = \triangle t_0 t_1 t_2 \to E^d; \quad F(u) = \sum B_{ijk}^{\triangle}(u) b_{ijk} \\ G &: \tilde{\triangle} = \triangle t_0 \tilde{t}_1 \tilde{t}_2 \to E^d; \quad G(u) = \sum B_{ijk}^{\tilde{\triangle}}(u) \tilde{b}_{ijk} \end{aligned}$$

Theorem 3.7 *Let* $F, G, \lambda_0, \lambda_1, \lambda_2$ *are defined like in the Situation 1. Let*

$$\begin{aligned} b_{ijk}^0 &= b_{ijk} \\ b_{ijk}^l &= \lambda_0 b_{i+1jk}^{l-1} + \lambda_1 b_{ij+1k}^{l-1} + \lambda_2 b_{ijk+1}^{l-1}; \quad l = 1, \dots n, \quad i+j+k+l = n \end{aligned}$$

Then F *and* G *are* C^q *smoothly joined along the common edge* $t_1 t_2$ *if and only if*

$$\tilde{b}_{ijk} = b_{0jk}^i, \quad i = 0, \dots q; \ i+j+k = n$$

Proof: First, using the mathematical induction we will show, that

$$b_{ijk}^l = f(t_0^i t_1^j t_2^k \tilde{t}_0^l). \tag{7}$$

1° when $l = 0$ then $b_{ijk}^0 = b_{ijk} = f(t_0^i t_1^j t_2^k)$ according to the (3).
2° let (7) holds for $l = l' - 1$. Then

$$\begin{aligned} b_{ijk}^{l'} &= \lambda_0 b_{i+1jk}^{l'-1} + \lambda_1 b_{ij+1k}^{l'-1} + \lambda_2 b_{ijk+1}^{l'-1} = \\ &= \lambda_0 f(t_0^{i+1} t_1^j t_2^k \tilde{t}_0^{l'-1}) + \lambda_1 f(t_0^i t_1^{j+1} t_2^k \tilde{t}_0^{l'-1}) + \lambda_2 f(t_0^i t_1^j t_2^{k+1} \tilde{t}_0^{l'-1}) = \\ &= f((\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2) t_0^i t_1^j t_2^k \tilde{t}_0^{l'-1}) = \\ &= f(t_0^i t_1^j t_2^k \tilde{t}_0^{l'}) \end{aligned}$$

And now, it is possible to prove the assertion of the theorem. It is very simple and we will not do it here in details. To prove both necessary and sufficient condition we need to use, besides the just proved property, also Theorem 3.5, multiaffinity of polar forms and the fact that univariable Bernstein polynomials of degree n create a basis of all polynomial functions of degree up to n. ■

Theorem 3.8 *Let $\triangle, \tilde{\triangle}$, $\lambda_{ij}; i = 1, 2; j = 0, 1, 2;$ F, G are defined as in the situation 2. Let*

$$\begin{aligned} b^{00}_{ijk} &= b_{ijk} \\ b^{l0}_{ijk} &= \lambda_{10} b^{l-10}_{i+1jk} + \lambda_{11} b^{l-10}_{ij+1k} + \lambda_{12} b^{l-1m}_{ijk+1}; \quad l = 1, \dots n,\ i + j + k + l = n \\ b^{lm}_{ijk} &= \lambda_{20} b^{lm-1}_{i+1jk} + \lambda_{21} b^{lm-1}_{ij+1k} + \lambda_{22} b^{lm-1}_{ijk+1}; \quad m = 1, \dots n - l,\ i + j + k + l + m = n \end{aligned}$$

Then F, G are C^q continuous at the vertex t_0 iff

$$\tilde{b}_{ijk} = b^{jk}_{i00} \quad \textit{for } j + k \leq q$$

Proof: The proof if very similar to the one of Theorem 3.7. By the induction we will show that

$$b^{lm}_{ijk} = f(t^i_0 t^j_1 t^k_2 \tilde{t}^l_1 \tilde{t}^m_2). \tag{8}$$

The induction step has to be done for both l and m. Then the necessary and sufficient conditions can to be proved. ■

3.2 Geometric Interpretation

Now, it is very simple to obtain a geometric interpretation of the continuity of several first orders. Let us have the notation from the Situation1. Then for the C^0 continuity it is really not hard to see that the following corollary follows.

Corollary 3.9 *Triangular Bézier patches $F : \triangle \to E^d, G : \tilde{\triangle} \to E^d$ are C^0 continuous along the common edge $t_1 t_2$ iff*

$$\tilde{b}_{0ij} = b_{0ij}; \quad i + j = n \tag{9}$$

i.e. the control vertices along their common edge are identical.

Corollary 3.10 *F and G are C^1 smoothly joined along the common edge iff they are C^0 continuous there and, moreover, the control points $b_{1ij}, b_{0i+1j}, \tilde{b}_{1ij}, b_{0ij+1}; i + j + k = n$ are coplanar and create an affine image of the quadrangle $t_0 t_1 \tilde{t}_0 t_2$, i.e.*

$$\tilde{b}_{1ij} = \lambda_0 b_{1ij} + \lambda_1 b_{0i+1j} + \lambda_2 b_{0ij+1}; \quad i + j = n - 1 \tag{10}$$

Proof: The assertion follows directly from Theorem 3.7 : $\tilde{b}_{1jk} = b^1_{0jk} = \lambda_0 b_{1ij} + \lambda_1 b_{0j+1k} + \lambda_2 b_{0jk+1}$

Corollary 3.11 *Bézier patches $F : \triangle \to E^d, G : \tilde{\triangle} \to E^d$ are C^2 smoothly joined along the common boundary iff they are C^1 continuous there and moreover there exist points $d_j;\ j = 1, \dots n-1$ such that vertices $b_{2jk}, b_{1j+1k}, d_j, b_{1jk+1}$ are coplanar and create an affine image of the quadrangle $t_0 t_1 \tilde{t}_0 t_2$ and also points $d_j, \tilde{b}_{1j+1k}, \tilde{b}_{2jk}, \tilde{b}_{1jk+1}$ are coplanar and create an affine image of the quadrangle $t_0 t_1 \tilde{t}_0 t_2$ too.*

Proof: From Theorem 3.7 :

$$\begin{aligned} \tilde{b}_{2jk} &= b^2_{0jk} = \lambda_0 b^1_{1jk} + \lambda_1 b^1_{0j+1k} + \lambda_2 b^1_{0jk+1} = \\ &= \lambda_0 d_j + \lambda_1 \tilde{b}_{1j+1k} + \lambda_2 \tilde{b}_{1jk+1}, \end{aligned}$$

where $d_j = \lambda_0 b_{2jk} + \lambda_1 b_{1j+1k} + \lambda_2 b_{1jk+1}$. ■

The last corollary treats with the continuity at the vertex.

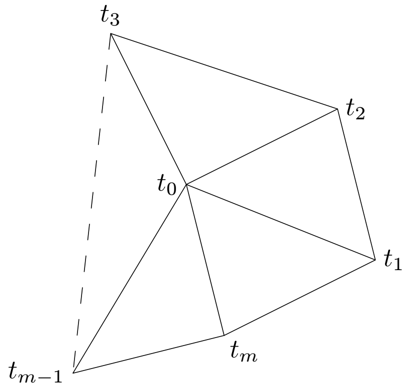

Figure 2: The star of the vertex t_0.

Corollary 3.12 *Let domains $\triangle, \tilde{\triangle}$ and the patches F, G are defined as in the Situation 2. F, G are C^1 smoothly joint in t_0 iff $b_{n00} = \tilde{b}_{n00}$ (C^0 condition) and control points $b_{n00}, b_{n-1,1,0}, b_{n-1,0,1}, \tilde{b}_{n-1,1,0}, \tilde{b}_{n-1,0,1}$ create an affine image of the vertices $t_0, t_1, t_2, \tilde{t}_0, \tilde{t}_1$.*
Moreover, if vertices $t_0, t_1, t_2, \ldots t_n$ create a star of t_0 and patches $F_i;\ i = 1, 2, \ldots m$ are defined such that $\triangle t_0 t_i t_{i+1}$ is a domain of $F_i; i = 1, \ldots n$ (indices continue cyclicaly: t_{m+1} means t_1), see Figure 2, then $F_i; i = 1, \ldots n$ are C^1 smoothly joint in t_0 iff

$$b_{i,n00} \text{ are equal for } i = 1, \ldots m \ (C^0 \text{ continuity condition})$$
$$b_{i+1,n-1,1,0} = b_{i,n-1,0,1};\ i = 1, \ldots m$$
$$b_{1,n00} b_{1,n-1,1,0} b_{2,n-1,1,0} \ldots b_{m,n-1,1,0} \text{ create an affine image of } t_0 t_1 t_2 \ldots t_m$$

where $b_{l,ijk}$ are control points of F_l : $b_{l,ijk} = f_l(t_0^i t_l^j t_{l+1}^k)$

Proof: The proof of the first part is very similar to the one of Corollary 3.10. The second part is the direct corollary of the first part.

$$\begin{aligned}
\tilde{b}_{n-1,1,0} &= b^{10}_{n-1,0,0} = \lambda_{10} b_{n00} + \lambda_{11} b_{n-1,1,0} + \lambda_{12} b_{n-1,0,1} \\
\tilde{b}_{n-1,0,1} &= b^{01}_{n-1,0,0} = \lambda_{20} b_{n00} + \lambda_{21} b_{n-1,1,0} + \lambda_{22} b_{n-1,0,1}
\end{aligned}$$

i.e. $b_{n00} b_{n-1,1,0} b_{n-1,0,1} \tilde{b}_{n-1,1,0} \tilde{b}_{n-1,0,1}$ is an affine image of the $t_0 t_1 t_2 \tilde{t}_1 \tilde{t}_2$. ■

Today geometric interpretations of the continuity of the higher degrees are known. The complete description of them can be found e.g. in [Kas98].

4 Description of Domain

In the following we will construct bivariate spline maps. They are piecewise polynomial maps from a subset Ω of two-dimensional manifold into R. Ω will be divided into finite number of triangles and each triangle will be associated with a polynomial map.

The domain Ω of bivariate spline functions can be bounded polygon, bounded polygon with polygonal holes or it can be considered closed without any border (connected like polyhedron or polyhedral tore etc.).

The denotation in the following definition is valid for the rest of the work:

Definition 4.1 *The domain Ω is the subset of 2-manifold.* ***The triangulation*** *$\mathcal{T}$ of Ω is the set of all vertices, edges and triangles of the triangulated domain. V is the number of vertices,*

E is the number of edges and F is the number of triangles in the triangulation $\mathcal{T}$. Further notation is:

N – the number of boundaries (holes),
V_B – number of vertices of all boundaries,
V_I – number of interior vertices,
E_B – number of edges from all boundaries,
E_I – number of interior edges.

In all the work we will consider only regular triangulations, where all triangles are proper, i.e. the vertices of each triangle are non-collinear.

Lemma 4.2 *Let Ω is triangulated (by the triangulation $\mathcal{T}$) domain without any border. Then*

$$E = 3(V-2); \quad F = 2(V-2)$$

Proof:
1° if $V = 4$ then they are connected each other and we have $E = 6$ and $F = 4$ – the domain is in fact a tetrahedron.
2° first, let us increase the number of vertices. For $V-1$ vertices according to the induction hypothesis $E = 3(V-3)$ and $F = 2(V-3)$. By adding a vertex into the arbitrary face we obtain 3 new edges and 2 new triangles, so that for V vertices there are $E = 3(V-3)+3 = 3(V-2)$ edges and $F = 2(V-3)+2 = 2(V-2)$ triangles.

And now, let us increase the genus of the domain. Let $\mathcal{T}$ has V vertices, $E = 3(V-2)$ edges and $T = 2(V-2)$ triangles. Let T_1, T_2 are triangles from $\mathcal{T}$ with no common vertex – we make them equal and left their faces. The genus increases and there are $V' = V-3$ vertices, $E' = E-3$ edges and $F' = F-2$ faces in the triangulation of the new domain. The equations $E' = 3(V-2),\ F' = 2(V-2)$ stay satisfied. ■

Lemma 4.3 *Let Ω is the triangulated (by the triangulation $\mathcal{T}$) domain with N boundaries (holes) and let V_B denotes number of vertices creating all boundaries. Then*

$$E = 3V - V_B + 3(N-2); \quad F = 2V - V_B + 2(N-2)$$

Proof: First let us consider the domain Ω with just one boundary. When we consider it to be a polygonal hole, we can triangulate it and then obtain a triangulated domain without any boundary. It is very simple to convince oneself that by triangulation of a polygon by $V_B - 3$ edges $V_B - 2$ new faces are obtained.
Now, let Ω has N boundaries with $V_1, V_2, \ldots, V_N$ vertices; $V_1 + \ldots + V_N = V_B$. By triangulating each hole we gain $V_i - 3$ new edges and $V_i - 2$ new faces for $i = 1 \ldots N$ and obtain a domain without any boundary. Therefore for Ω it holds :

$$\begin{aligned} E &= 3(V-2) - \sum_{i=1}^{N}(V_i - 3) = 3V - V_B + 3(N-2) \\ F &= 2(V-2) - \sum_{i=1}^{N}(V_i - 2) = 2V - V_B + 2(N-2) \end{aligned}$$

■

Note, that Lemma 4.2 is only a special case of Lemma 4.3

5 Intermezzo

There is known modeling of smooth piecewise polynomial surfaces over arbitrary triangulation using the bivariate B-splines. It is introduced for example in [Sei92]. Let us try to focus only on a special case: the modeling of C^0 continuous surfaces of the first degree (in this section we will call them piecewise linear).

Let t_0, t_1, t_2, t_3 are fixed points from E^2 (Euclidean 2-dimensional space), $u \in E^2$ is an arbitrary. Then

$$\begin{aligned} M_0(u) &= \frac{\chi_{[t_1 t_2 t_3)}}{d(\triangle t_1 t_2 t_3)} \\ M_1(u) &= \frac{\chi_{[t_0 t_2 t_3)}}{d(\triangle t_0 t_2 t_3)} \\ M_2(u) &= \frac{\chi_{[t_0 t_1 t_3)}}{d(\triangle t_0 t_1 t_3)}, \end{aligned}$$

where χ_A is the characteristic function of the set A and where $d[\triangle t_i t_j t_k)$ is the non-signed area of the triangle $\triangle t_i t_j t_k$ and $\chi_{[t_i t_j t_k)}$ is a characteristic function on the half-open convex hull $[t_i t_j t_k)$, which is an interior of the triangle $\triangle t_i t_j t_k$ with a part of its boundary (for exact definition of the half-open convex hull see e.g. [Sei92]). If $\lambda_i, \lambda_j, \lambda_k$ are such real numbers that $u = \lambda_i t_i + \lambda_j t_j + \lambda_k t_k$ then

$$M(u) = \lambda_i M_i(u) + \lambda_j M_j(u) + \lambda_k M_k(u),$$

where i, j, k are different numbers from $0, 1, 2, 3$. M is C^0 continuous bivariate B-spline of the first degree over the points t_0, t_1, t_2, t_3.

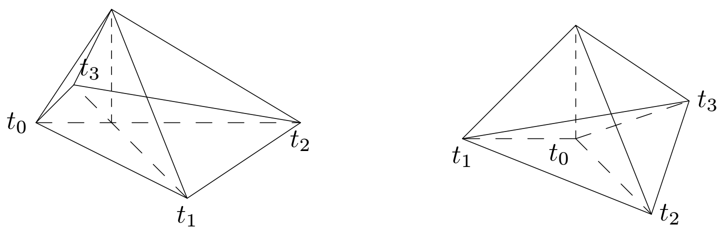

Figure 3: Two C^0 continuous B-spline functions over the two possible configurations of points t_0, t_1, t_2, t_3. The graph of each is a pyramid with the base $t_0 t_1 t_2 t_3$ resp. $t_0 t_1 t_2$.

B-spline function as defined above has the following continuity property: if no triplet of points from t_0, t_1, t_2, t_3 are collinear then M is C^0 continuous along each edge $[t_i, t_j]$; $i, j = 0, 1, 2, 3$ and everywhere else it is polynomial (i.e. C^∞ continuous). If e.g. t_0, t_1, t_2 are collinear then along this edge M is discontinuous.

The construction of C^0 continuous surfaces above the arbitrary given triangulation using B-splines is provided as follows (see [Sei92]): for each vertex t_i from the triangulation the new one is added, let us denote it $\tilde{t}_i$. Then above each triangle $\triangle t_i t_j t_k \in \mathcal{T}$ three B-spline functions are constructed: for the quadruples $t_i t_j t_k \tilde{t}_i$, $t_i t_j t_k \tilde{t}_j$, $t_i t_j t_k \tilde{t}_k$. The consequent surface is a linear combination of all this B-splines functions.

Let us consider the given triangulation consisting only of the only triangle $\triangle t_0 t_1 t_2$ and the consequent edges and vertices. Let us add the vertices $\tilde{t}_0$, $\tilde{t}_1$, $\tilde{t}_2$ such that no triplet $t_i, t_j, \tilde{t}_k$; $i, j, k \in 0, 1, 2$, $i \neq j$ of them is collinear to save the C^0 continuity of the consequent surface. Then according the [Sei92] we will construct three B-splines above the quadruples $t_i t_j t_k \tilde{t}_i$, $t_i t_j t_k \tilde{t}_j$, $t_i t_j t_k \tilde{t}_k$. The surface which arises as a linear combination of these three B-splines is C^0 continuous along each of the following edges: $t_i t_j$, $\tilde{t}_i t_i$, $\tilde{t}_i t_j$; $i, j, = 0, 1, 2$, $i \neq j$, but it is not inevitably C^∞ continuous there. The original input triangulation has remarkable refined and the C^0 continuous surface is constructed over this refined tesselation.

Only for the interest we can compute how many new edges are added:

For each given vertex t_i we add $\widetilde{t_i}$ and we connect this new vertex with t_i and with all such vertices of the original triangulation, that t_i is their neighbour (they have a common edge). So

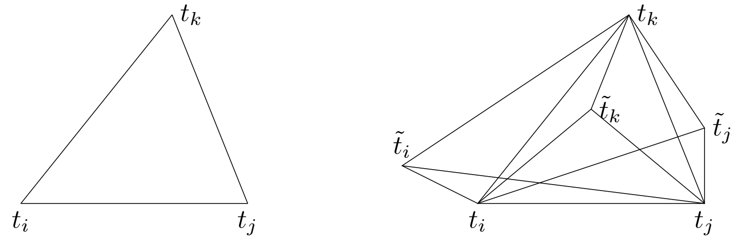

Figure 4: The original triangulation (left) and its refinment (right).

we add at most

$$V + \sum_{i=1}^{V} \deg t_i = V + 2E$$

new edges (in case of colinearity it is less) where V is the number of vertices and E is number of edges in the given triangulation.

But the described way of modeling allows to construct an arbitrary C^0 continuous piecewise affine surface exactly over the original triangulation (not refined), if the additional vertices $\widetilde{t}_i$ are suitable chosen:

Let

$$f(u) = \sum_{i=1}^{V} \sum_{j=1}^{m_i} d_j^i B_j^i(u)$$

where V is the number of vertices in the original triangulation, m_i is the degree of the i-th vertex and $B_1^i(u), \ldots, \mathcal{B}_{m_i}^i(u)$ are all bivariate B-spline functions that $\widetilde{t}_i$ is one of the vertices of quadruple, they are defined over. Then if

$$\widetilde{t}_i = t_i \tag{11}$$

$$d_1^i = d_2^i = \ldots = d_m^i \tag{12}$$

then f is C^0 continuous surface of the first degree exactly over the given triangulation (not refined). The reason is very simple.

The condition (11) implies that $B_j^i(u);\ i = 1, \ldots, V,\ j = 1, \ldots, m_i$ is a Bézier patch of the first degree (see [Sei92]). There are three Bézier patches defined over each triangle from the given triangulation. Then d_j^i are Bézier control points of these patches over t_i. The second condition (12) satisfies C^0 continuity at t_i. Since we are constructing the function of first degree, the C^0 continuity at all vertices fromn the triangulation induces the C^0 continuity along edges.

Finally, we are introducing the second way of modeling C^0 continuous piecewise linear surfaces.

The role of the elementary functions will not be played by bivariate B-splines. They will be substituted by the following functions:

$$f_i(u) = \begin{cases} 1 & \text{if } u = t_i \\ \frac{d(ut_j)}{d(t_it_j)} & \text{if } [t_i, t_j] \in \mathcal{T} \text{ and } u \in [t_i, t_j] \setminus \{t_i, t_j\} \\ \frac{d(\triangle ut_jt_k)}{d(\triangle t_it_jt_k)} & \text{if } \triangle t_i, t_j, t_k \in \mathcal{T} \text{ and } u \in \text{ int } \triangle t_it_jt_k \\ 0 & \text{otherwise} \end{cases}$$

where $d(t_it_j)$ is the distance of t_i and t_j.

We can simply convince ourselves that f_i is C^0 continuous for all $t_i \in \mathcal{T}$. First, we will show that f_i is C^0 continuous in t_i.

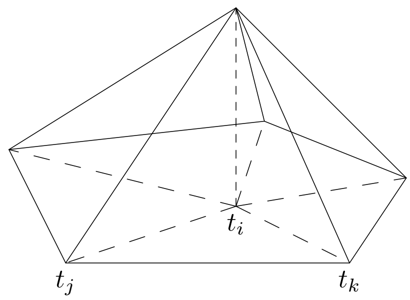

Figure 5: C^0 continuous elementary functions of the first degree defined over the star of the vertex t_i. Its graph is a pyramid, its base is the star of t_i.

Let $u \in [t_i, t_j]$, where $[t_i, t_j] \in \mathcal{T}$, $u \neq t_j$. Then

$$\lim_{u \to t_j} f_i(u) = \lim_{u \to t_j} \frac{d(ut_j)}{d(t_i t_j)} = \frac{d(t_j t_j)}{d(t_i t_j)} = 0$$

Let $u \in \text{int} \triangle t_i t_j t_k$, where $\triangle t_i t_j t_k \in \mathcal{T}$. Then

$$\lim_{u \to t_j} f_i(u) = \lim_{u \to t_j} \frac{d(\triangle u t_j t_k)}{d(\triangle t_i t_j t_k)} = \frac{d(\triangle t_j t_j t_k)}{d(\triangle t_i t_j t_k)} = 0$$

If u does not belong to any triangle incidenting with t_i, then $f_i(u) = 0$. It implies that f_i is C^0 continuous in t_j.

Similarly it can be shown that f_i is continuous in t_i, in each point of $[t_i, t_j]$, where $[t_i, t_j] \in \mathcal{T}$ and in each point of $[t_j, t_k]$, where $\triangle t_i t_j t_k \in \mathcal{T}$.

The defined functions f_i for all vertices t_i have besides the C^0 continuity also the second important property. Each C^0 continuous piecewise linear function defined over $\mathcal{T}$ can be expressed as a sum

$$g = \sum_{i=1}^{V} d_i f_i.$$

Really, for each f_i holds $f_i(t_i) = 1$ and $f_i(t_j) = 0$ for $t_i \neq t_j$. Further, each piecewise linear function g is uniquely determined by its values in the vertices from $\mathcal{T}$ $g(t_i) = g_i$. Therefore g can be expressed as $g = \sum g_i f_i$. Moreover, each piecewise linear function can be such expressed by the only way. It means that $\{f_i; i = 1, \ldots, V\}$ is the basis of the space of all piecewise linear functions over the triangulation $\mathcal{T}$.

Note, that this method of modeling does not refine the original triangulation.

Naturally the question arises if it is possible to construct also the surfaces of the higher order of continuity and of the higher degree similarly. The work tries to find an answer.

6 Dimension of $S_n^1(\mathcal{T})$

Definition 6.1 *Let Ω is the domain with the triangulation $\mathcal{T}$, n is the degree.*

$$P_n := span\{x^i y^j; i \geq 0, j \geq 0, i + j \leq n\}$$

*is the **space of bivariate polynomials**, $P_n : \Omega \to R$. Let q is the order of the continuity, $0 \leq q < n$ and $C^q(\Omega)$ be the **space of all functions with all partial derivatives up to q continuous**.*

$$S_n^q(\mathcal{T}) := \{F \in C^q(\Omega); F|_\triangle \in P_n \; \forall \triangle \in \mathcal{T}\}$$

is the ***space of*** C^q ***continuous bivariate polynomial spline functions of degree*** n ***defined over*** $\boldsymbol{\Omega}$.

It is obvious that $S_n^q(\mathcal{T})$ is the linear space. The element of the space is piecewise polynomial function, which is composed of triangular Bézier patches; for each triangle $\triangle \in \mathcal{T}$ there is defined a Bézier patch.

Let Ω is the given triangulated domain. We will study the space of C^1 continuous bivariate splines of n-th degree, it is the space $S_n^1(\mathcal{T})$.

Each C^1 bivariate polynomial is one-to-one determined by its control net created by control nets of all Bézier triangular patches. Let us consider from the very start that our splines are already C^0 continuous so that according (9) the control points of two neighbouring patches over the common edge are the same. For all identical points, which are identical because of C^0 continuity, we will take only one of them to the control net of the spline.

The number of control points is then

$$\#CP = V + (n-1)E + \frac{(n-1)(n-2)}{2}F \tag{13}$$

Because of the C^1 continuity there are n quadruples of points along each interior edge which must keep the condition (10) We has obtained a system of $n.E_I$ linear homogeneous equations (E_I denotes the number of interior edges). Let $\boldsymbol{A}$ is the matrix of this system. Then $F \in S_n^1(\mathcal{T})$ if and only if the control points of F satisfy the system determined by the matrix $\boldsymbol{A}$. It means that $S_n^1(\mathcal{T})$ is exactly its solution space. If $r(\boldsymbol{A})$ is the rank of the matrix $\boldsymbol{A}$, then for the dimension of $S_n^1(\mathcal{T})$ holds

$$dim(S_n^1(\mathcal{T})) = \#CP - r(\boldsymbol{A})$$

Note 6.2 *Sometimes, instead of "the system of equations determined by* $\boldsymbol{A}$*" we will use simply only "*$\boldsymbol{A}$*"; for example "to satisfy* $\boldsymbol{A}$*" means "to satisfy the system of equations determined by* $\boldsymbol{A}$*" and so on.*

Example. Let Ω looks like in the Situation 1 (see Figure 1, left). Then $\mathcal{T}$ consists of two triangles and contains only one interior edge t_1t_2. Let degree $n = 3$. Then the system of equations looks

$$\begin{aligned}
\widetilde{b}_{120} &= \lambda_0 b_{120} + \lambda_1 b_{030} + \lambda_2 b_{021} \\
\widetilde{b}_{111} &= \lambda_0 b_{111} + \lambda_1 b_{021} + \lambda_2 b_{012} \\
\widetilde{b}_{102} &= \lambda_0 b_{102} + \lambda_1 b_{012} + \lambda_2 b_{003}
\end{aligned}$$

The matrix of this system is

$$\begin{pmatrix}
-1 & 0 & 0 & \lambda_0 & 0 & 0 & \lambda_1 & \lambda_2 & 0 & 0 \\
0 & -1 & 0 & 0 & \lambda_0 & 0 & 0 & \lambda_1 & \lambda_2 & 0 \\
0 & 0 & -1 & 0 & 0 & \lambda_0 & 0 & 0 & \lambda_1 & \lambda_2
\end{pmatrix}$$

Obviously $r(\boldsymbol{A}) = 3$. The number of control points is $10 + 6 = 16$. It means that for this Ω it holds : $dim(S_3^1(\mathcal{T})) = 16 - 3 = 13$.

How to model the splines in $S_n^1(\mathcal{T})$? We need to find a solution of the system determined by $\boldsymbol{A}$ to have found a spline function from $S_n^1(\mathcal{T})$. It can be done by determining of $dim(S_n^1(\mathcal{T}))$ independent values of control points and calculating the others to satisfy $\boldsymbol{A}$. The great disadvantage of this method is its complexity, both the time and the space. We need to solve the matrix $\boldsymbol{A}$ which can be really huge.

The second way of modeling of spline functions over the triangulation $\mathcal{T}$ was already briefly featured. If $E_i;\ i = 1, \dots, dim(S_4^1(\mathcal{T}))$ is the basis of $S_4^1(\mathcal{T})$, then each bivariate

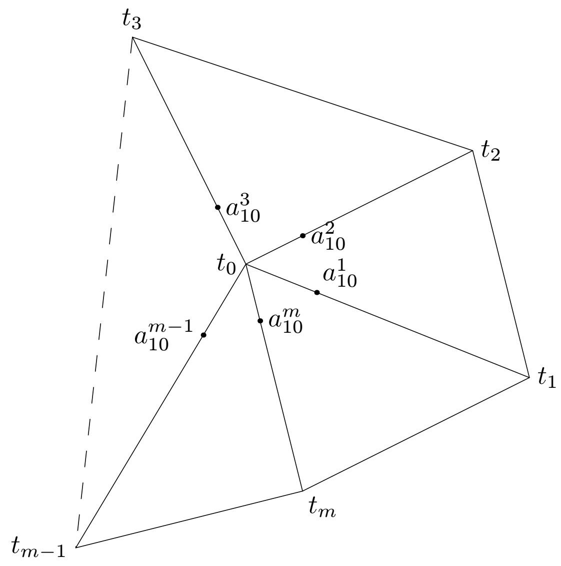

Figure 6: The star of t_0 with abscissae $a_{10}^1 \dots a_{10}^m$.

spline function from this space can be uniquely expressed as a linear combination of the basis functions.

We want to model functions from $S_4^1(\mathcal{T})$ using the second method. Therefore we need to know a basis of $S_4^1(\mathcal{T})$. The aim of this section is to compute $dim(S_4^1(\mathcal{T}))$. The task is equivalent to the computing the $r(S_4^1(\mathcal{T}))$ and $\#CP$. In the next section we will use the result to finding the basis of $S_4^1(\mathcal{T})$.

Note 6.3 *The notion of equivalence of matrices will be used in an extended meaning.* $\boldsymbol{M_1} \sim \boldsymbol{M_2}$ *if there exist a sequence of equivalent reductions leading from* $\boldsymbol{M_1}$ *to* $\boldsymbol{M_2}$*. But* $\boldsymbol{M_1} \sim \boldsymbol{M_2}$ *even if* $\boldsymbol{M_2}$ *has raised from* $\boldsymbol{M_1}$ *by adding or omitting a zero row or a row linearly dependent on the others. Similarly a zero column or a linearly dependent column can be added or omitted.*

For each interior vertex of Ω we can describe its star (see Figure 6):

– m is degree of t_0

– $t_1, \dots, t_m$ are neighbouring vertices to t_0

– $\triangle_1, \dots, \triangle_m$ are such triangles, that one of the vertices of $\triangle_i$ is t_0: $\triangle_i = \triangle t_0 t_i t_{i+1}$

– a_{10}^i are specially chosen Greville abscissa of the patch F_i; F_i if defined on $\triangle_i$; $i = 1, \dots, m$. a_{10}^i is such abscissa, that $a_{10}^i = \frac{1}{n}((n-1)t_0 + t_i)$ (see Lemma 2.10) and b_{10}^i is corresponding control points of F_i, therefore $b_{10}^i = f_i(t_0^{n-1} t_i)$, where f_i is the polar form of F_i (see Lemma 2.9).

Moreover b_{00} is the control point associated to t_0: $b_{00} = f(t_0^n) = F(t_0)$. For a boundary vertex (i.e. one of the vertices of a boundary) the triangle $\triangle_m$ does not exist, but abscissa a_{10}^m does exist and b_{10}^m is defined. They belong to $\triangle_{m-1}$. The polygonal line $t_m t_0 t_1$ is a part of a boundary.

Be careful about the meaning of the notation of control points, it is different from the previous one. The upper index gives an information on the patch the control point belongs to.

Let $\lambda_{i0}, \lambda_{i1}, \lambda_{i2},\ i = 1, \dots m$ are barycentric coordinates of t_{i+1} w. r. t $\triangle_{i-1}$: $t_{i+1} = \lambda_{i0} t_0 + \lambda_{i1} t_{i-1} + \lambda_{i2} t_i$. All indices are cyclic in sequence $(1, \dots m)$; for example t_{i+1} for $i = m$ means t_1 and t_{i-1} for $i = 1$ means t_m.

Let us permute the rows of the matrix $\boldsymbol{A}$ to get V groups of conditions (one group around each vertex) to the top rows. It is possible to do so for $n \geq 2$. (For $n = 1$ each conditions occurs in two groups and it is not possible to reach such situation by permuting the rows.) The group for interior vertex t_0 on Figure 6 will consist of these equations:

$$b_{10}^{i+1} = \lambda_{i0} b_{00} + \lambda_{i1} b_{10}^{i-1} + \lambda_{i2} b_{10}^{i};\ i = 1, \ldots m \tag{14}$$

There are m equations with $m+1$ unknowns $b_{10}^1, \ldots b_{10}^m, b_{00}$ and they form a submatrix $\boldsymbol{A_1}$ of $\boldsymbol{A}$:

$$\boldsymbol{A_1} = \begin{pmatrix} \lambda_{12} & -1 & 0 & 0 & \ldots & \lambda_{11} & \lambda_{10} \\ \lambda_{21} & \lambda_{22} & -1 & 0 & \ldots & 0 & \lambda_{20} \\ 0 & \lambda_{31} & \lambda_{32} & -1 & \ldots & 0 & \lambda_{30} \\ \ldots & & & & & & \\ -1 & 0 & 0 & 0 & \ldots & \lambda_{m2} & \lambda_{m0} \end{pmatrix}$$

If t_0 is a boundary vertex, then there are only $m-2$ equations from (14) – for $i = 2, \ldots, m-1$ and also the according matrix $\boldsymbol{A_1}$ is different:

$$\boldsymbol{A_1} = \begin{pmatrix} \lambda_{21} & \lambda_{22} & -1 & \ldots & 0 & \lambda_{20} \\ 0 & \lambda_{31} & \lambda_{32} & \ldots & 0 & \lambda_{30} \\ \ldots & & & & & \\ 0 & 0 & 0 & \ldots & -1 & \lambda_{m-10} \end{pmatrix}$$

Lemma 6.4 $r(\boldsymbol{A_1}) = m - 2$.

Proof: If $\boldsymbol{A_1}$ is the matrix belonging to a boundary vertex, then $\boldsymbol{A_1}$ is triangular with $m-2$ rows so that assertion follows since $\lambda_{n1} < 0$. For interior vertex equations of the matrix $\boldsymbol{A_1}$ exactly say, that patches $F_1, \ldots, F_m$ are C^1 smoothly joint in t_0. According to the Corollary 3.12 the points $b_{00}, b_{10}^1, \ldots b_{10}^m$ create an affine image of the vertices $t_0, t_1, \ldots t_m$. So there exists a bijection between the solution space of $\boldsymbol{A_1}$ and the set of all affine functions defined over $\{t_0, a_{10}^1, \ldots, a_{10}^m\}$. This set is also the linear space and its dimension is 3. Therefore the dimension of the solution space of the system (14) is 3 and the assumption follows. ■

Note 6.5 *It is simple to get convinced, that if we omit two arbitrary sequential rows or first and the last one from (14) we'll obtain a regular matrix.*

Now, let us add new abscissae into our figure:
a_{20}^i, a_{11}^i are abscissae of F_i such that

$$a_{20}^i = \frac{1}{n}((n-2)t_0 + 2t_i), \quad a_{11}^i = \frac{1}{n}((n-2)t_0 + t_i + t_{i+1})$$

and again b_{20}^i, b_{11}^i are such control points that

$$b_{20}^i = f_i(t_0^{n-2} t_i^2) \text{ and } b_{11}^i = f_i(t_0^{n-2} t_i t_{i+1}) \text{ for } i = 1, \ldots, m$$

If t_0 is a boundary vertex, then a_{11}^m, b_{11}^m do not exist.

Let us make further permutation of rows of $\boldsymbol{A}$ to extend the groups of conditions for control points. These new equations for the vertex t_0 on Figure 7 are:

$$\begin{aligned} b_{11}^i = \lambda_{i0} b_{10}^i + \lambda_{i1} b_{11}^{i-1} + \lambda_{i2} b_{20}^i; \quad i &= 1, \ldots, m \text{ for an interior vertex} && (15) \\ i &= 2, \ldots, m-1 \text{ for a boundary vertex} && (16) \end{aligned}$$

i.e. $b_{10}^i, b_{11}^{i-1}, b_{20}^i, b_{11}^i$ form an affine image of $t_0, t_{i-1}, t_i, t_{i+1}$. There are m resp. $m-2$ new equations with $3m+1$ resp. $3m$ unknowns, $2m$ resp. $2m-1$ of them are new. It is possible to do so for $n \geq 4$, because then there are at least 4 quadruples of points along each edge. (For

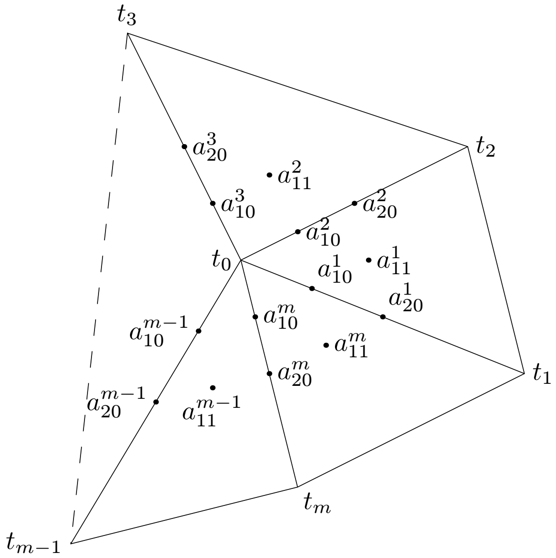

Figure 7: The star of t_0 with abscissae $a_{10}^1 \dots a_{10}^m$, $a_{20}^1 \dots a_{20}^m$ and $a_{11}^1 \dots a_{11}^m$.

smaller n there occurs a condition which is contained in two groups.) The submatrix $\boldsymbol{A_2}$ of $\boldsymbol{A}$ determined by (14) and (15) consists of several blocks:

$$\boldsymbol{A_2} = \begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{B_0} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A_1} \end{pmatrix}$$

The columns of $\boldsymbol{B_1}$ belong to respective unknowns $b_{11}^1, b_{11}^2, \dots, b_{11}^m$:

$$\boldsymbol{B_1} = \begin{pmatrix} -1 & 0 & 0 & \dots & \lambda_{11} \\ \lambda_{21} & -1 & 0 & \dots & 0 \\ 0 & \lambda_{31} & -1 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}$$

for the interior vertex; for the boundary one without the first and the last row and the last column (control point b_{11}^m does not exist).

The columns of $\boldsymbol{B_2}$ belong to respective unknowns $b_{20}^1, b_{20}^2, \dots, b_{20}^m$:

$$\boldsymbol{B_2} = \begin{pmatrix} \lambda_{12} & 0 & 0 & \dots & 0 \\ 0 & \lambda_{22} & 0 & \dots & 0 \\ 0 & 0 & \lambda_{32} & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & \lambda_{m2} \end{pmatrix}$$

for the interior vertex; for the boundary one without the first and the last row.

The columns of $\boldsymbol{B_0}$ belong to respective unknowns $b_{10}^1, b_{10}^2, \dots, b_{10}^m, b_{00}$:

$$\boldsymbol{B_0} = \begin{pmatrix} \lambda_{10} & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_{20} & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_{30} & \dots & 0 & 0 \\ \dots \\ 0 & 0 & 0 & \dots & \lambda_{m0} & 0 \end{pmatrix}$$

for the interior vertex; for the boundary one without the first and the last row.

$\boldsymbol{O}$ is a zero matrix of the type $m \times m$ for interior vertex resp. $(m-2) \times m$ or $(m-2) \times (m-1)$ for the boundary one.

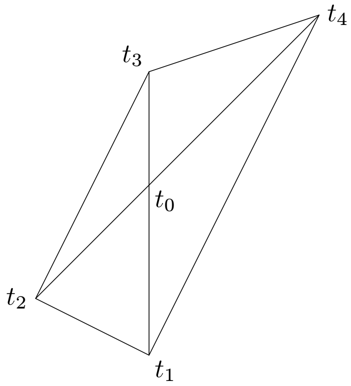

Figure 8: Singular vertex.

The following lemma gives us the answer to the question about the rank of $\boldsymbol{A_2}$. To answer it, we need the notion of singular vertex.

Definition 6.6 ***Singular vertex*** *is such an interior vertex that its degree $m = 4$ and adjacent edges form two diagonals of a quadrangle. V_S denotes the number of singular vertices in the triangulation $\mathcal{T}$ (see Figure 8).*

Lemma 6.7 *Let $\boldsymbol{A_2}$ is defined as above. Then $r(\boldsymbol{A_2}) = 2m - 4$ iff t_0 is a boundary vertex, $r(\boldsymbol{A_2}) = 2m - 3$ iff t_0 is a singular vertex, and $r(\boldsymbol{A_2}) = 2m - 2$ otherwise.*

Proof: For the boundary vertex the matrix $\boldsymbol{A_2}$ is triangular and has $2m - 4$ rows so that $r(\boldsymbol{A_2}) = 2m - 4$.

Now, let us consider only interior vertices. Then

$$\boldsymbol{A_2} \sim \boldsymbol{A_3} = \begin{pmatrix} \boldsymbol{B_1'} & \boldsymbol{B_2'} & \boldsymbol{B_o'} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A_1'} \end{pmatrix},$$

where

$$\boldsymbol{B_1} \sim \boldsymbol{B_1'} = \begin{pmatrix} -1 & 0 & 0 & \dots & \lambda_{11} \\ 0 & -1 & 0 & \dots & \lambda_{11}\lambda_{21} \\ 0 & 0 & -1 & \dots & \lambda_{11}\lambda_{21}\lambda_{31} \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \Pi_1 - 1 \end{pmatrix},$$

$$\boldsymbol{B_2} \sim \boldsymbol{B_2'} = \begin{pmatrix} \lambda_{12} & 0 & 0 & \dots & 0 \\ \lambda_{12}\lambda_{21} & \lambda_{22} & 0 & \dots & 0 \\ \lambda_{12}\lambda_{21}\lambda_{31} & \lambda_{22}\lambda_{31} & \lambda_{32} & \dots & 0 \\ \dots & & & & \\ \lambda_{12}\Pi_2 & \lambda_{22}\Pi_3 & \lambda_{32}\Pi_4 & \dots & \lambda_{m2} \end{pmatrix},$$

$$\boldsymbol{B_0} \sim \boldsymbol{B_0'} = \begin{pmatrix} \lambda_{10} & 0 & 0 & \dots & 0 & 0 \\ \lambda_{10}\lambda_{21} & \lambda_{20} & 0 & \dots & 0 & 0 \\ \lambda_{10}\lambda_{21}\lambda_{31} & \lambda_{20}\lambda_{31} & \lambda_{30} & \dots & 0 & 0 \\ \dots & & & & & \\ \lambda_{10}\Pi_2 & \lambda_{20}\Pi_3 & \lambda_{30}\Pi_4 & \dots & \lambda_{m0} & 0 \end{pmatrix},$$

where $\Pi_i = \lambda_{i1}\lambda_{i+11}\dots\lambda_{m1}$ and $\boldsymbol{A_1'}$ has arised from $\boldsymbol{A_1}$ by omitting the first and the last row. $\boldsymbol{A_1'} \sim \boldsymbol{A_1}$ according to the Note 6.5. $\boldsymbol{A_1'}$ is triangular matrix with $m - 2$ rows.

Since $\lambda_{i0}, \lambda_{i1}, \lambda_{i2}$ are barycentric coordinates of t_{i+1} w. r. t. $\triangle_{i-1} = \triangle t_0 t_{i-1} t_i$, it holds that $\lambda_{i1} = -\frac{|\triangle_i|}{|\triangle_{i-1}|}$ where $|\triangle_i|$ is non-signed area of $\triangle_i$ (according to (2)) and therefore $\Pi_1 = (-1)^m$.

First, let us discuss the case that m is odd. Then in m-th row of $\boldsymbol{A_3}$ $\Pi_1 - 1 = -2$ and $\boldsymbol{B_1'}$ is triangular with m non-zero rows. Then whole matrix $\boldsymbol{A_3}$ is triangular and has $2m-2$ rows. It implies that $r(\boldsymbol{A_2}) = r(\boldsymbol{A_3}) = 2m-2$.

If m is even, the situation is more complicated. Because first $m-1$ rows are formed into a triangular matrix and the other rows has zero value on first m columns, the rank of $\boldsymbol{A_3}$ equals $(m-1)+r(\boldsymbol{A_4})$, where $\boldsymbol{A_4}$ contains just last $m-1$ rows of $\boldsymbol{A_3}$. But we can be already more precious. Since $r(\boldsymbol{A_1}) = m-2$, we can distinguish two cases :

1. $r(\boldsymbol{A_4}) = m-2$ iff its first row can be expressed as a linear combination of the others, then $r(\boldsymbol{A_2}) = 2m-3$ or
2. $r(\boldsymbol{A_4}) = m-1$ otherwise. Then $r(\boldsymbol{A_2}) = 2m-2$

It remains only to find out which kind of vertices satisfy condition of case 1 or 2.

Let us mention the first row of $\boldsymbol{A_4}$ – it consists of the last rows of $\boldsymbol{B_1'}, \boldsymbol{B_2'}, \boldsymbol{B_0'}$:

$(0\ 0\ \ldots 0 \quad \lambda_{12}\Pi_2\ \lambda_{22}\Pi_3\ \lambda_{32}\Pi_4\ \ldots\ \lambda_{m2} \quad \lambda_{10}\Pi_2\ \lambda_{20}\Pi_3\ \lambda_{30}\Pi_4\ \ldots\ \lambda_{m0} \quad 0)$

First, let us suppose, that the first row of $\boldsymbol{A_4}$ is a linear combination of the others. Since besides the first row the values of $(m+1)$-th to $(2m)$-th columns are zero, also the same columns of the first line has to be zero, thus:

$$\lambda_{i2}\Pi_{i+1} = 0; \quad i = 1, \ldots, m.$$

Since $\Pi_i = \lambda_{i1}\lambda_{i+11}\ldots\lambda_{m1}$ and all $\lambda_{i1} \neq 0$ (because $\lambda_{i1} = \frac{|\triangle_i|}{|\triangle_{i-1}|}$, surely $\Pi_i \neq 0$ for all $i = 1, \ldots, m$. It implies that $\lambda_{i2} = 0$ for each i. Because λ_{ij} are defined such that $t_{i+1} = \lambda_{i0}t_0 + \lambda_{i1}t_{i-1} + \lambda_{i2}t_i$, the fact that $\lambda_{i2} = 0$ geometrically means that t_{i-1}, t_0, t_{i+1} are collinear and t_0 lies between t_{i-1}, t_{i+1} since $\lambda_{i1} < 0$. To be more concrete: t_1, t_0, t_3 are collinear, t_2, t_0, t_4 are collinear and t_3, t_0, t_5 are again collinear. It forces $t_5 = t_1$ to have only proper triangles in the star. Figure 8 illustrates this situation. We can see, that t_0 is in fact singular vertex.

For interior vertices we have shown the necessary condition for holding $r(\boldsymbol{A_2}) = 2m - 3$. Now, we are going to prove, that being a singular vertex is also sufficient condition for $r(\boldsymbol{A_1}) = 2m-3$.

Since t_0 is singular according to assumption, from (2) we can write :

$$\lambda_{i0} = \frac{|\triangle_{i-1}| + |\triangle_i|}{|\triangle_{i-1}|}, \quad \lambda_{i1} = -\frac{|\triangle_i|}{|\triangle_{i-1}|}; \quad i = 1, \ldots, 4$$

and after assigning $\lambda_i = \frac{|\triangle_i|}{|\triangle_{i-1}|}$ we have

$$\lambda_{i1} = 1 + \lambda_i, \quad \lambda_{i0} = -\lambda_i; \quad i = 1, \ldots, 4$$

Now looking at Figure 8 we can say that

$$\begin{aligned} \frac{|t_1t_0|}{|t_3t_0|} &= \frac{|\triangle_1|}{|\triangle_2|} = \frac{1}{\lambda_2}; \quad \text{but also} \\ \frac{|t_1t_0|}{|t_3t_0|} &= \frac{|\triangle_4|}{|\triangle_3|} = \lambda_4 \end{aligned}$$

where $|t_it_0|$ is the length of the line t_it_0. It implies that $\lambda_2\lambda_4 = 1$. Similarly it can be shown that $\lambda_1\lambda_3 = 1$.

In this case, the matrix $\boldsymbol{A_4}$ has three rows: the first one is the last row of $\boldsymbol{B'_0}$ and the remaining two ones represent the matrix $\boldsymbol{A'_1}$ (it has $m-2=2$ rows):

$$\boldsymbol{A_4} = \begin{pmatrix} -(\lambda_1+1)\lambda_2\lambda_3\lambda_4 & (\lambda_2+1)\lambda_3\lambda_4 & -(\lambda_3+1)\lambda_4 & \lambda_4+1 & 0 \\ -\lambda_2 & 0 & -1 & 0 & \lambda_2+1 \\ 0 & -\lambda_3 & 0 & -1 & \lambda_3+1 \end{pmatrix}$$

Let us calculate this linear combination of the rows of $\boldsymbol{A_4}$ and show that it satisfies

$$l_1 - (\lambda_3+1)\lambda_4 l_2 + (\lambda_4+1)l_3 = 0$$

where l_i is i-th row of $\boldsymbol{A_4}$, which means that l_1, l_2, l_3 are linearly dependent. Really, it gives us a system of conditions put on the columns:

$$\begin{aligned} -(\lambda_1+1)\lambda_2\lambda_3\lambda_4 + (\lambda_3+1)\lambda_4\lambda_2 &= 0 \\ (\lambda_2+1)\lambda_3\lambda_4 - (\lambda_4+1)\lambda_3 &= 0 \\ -(\lambda_3+1)\lambda_4 + (\lambda_3+1)\lambda_4 &= 0 \\ \lambda_4+1-(\lambda_4+1) &= 0 \\ -(\lambda_2+1)(\lambda_3+1)\lambda_4 + (\lambda_4+1)(\lambda_3+1) &= 0 \end{aligned}$$

Which thanks to the fact that $\lambda_1\lambda_3 = \lambda_2\lambda_4 = 1$ are all always satisfied. The conclusion is that for singular vertex $r(\boldsymbol{A_4}) = 2 = m-2$ and $r(\boldsymbol{A_2}) = 2m-3$ and for other interior vertices $r(\boldsymbol{A_4}) = m-1$ and $r(\boldsymbol{A_2}) = 2m-2$. ■

Further we will treat with mentioned matrices for all vertices from $\mathcal{T}$ together, so we need to distinguish them.

Let $t_1, \ldots, t_V$ are all vertices from the triangulation $\mathcal{T}$. Let $\boldsymbol{A_{2i}}$ denotes the matrix $\boldsymbol{A_2}$ belonging to the vertex t_i; $i = 1, \ldots, V$. Let $\boldsymbol{A_{3i}}$ is the matrix $\boldsymbol{A_3}$ (mentioned in the proof of the previous Lemma) for t_i; $\boldsymbol{A_{3i}}$ consists of all linearly independent rows of $\boldsymbol{A_{2i}}$, so that $\boldsymbol{A_{3i}} \sim \boldsymbol{A_{2i}}$ and rank of $\boldsymbol{A_{3i}}$ equals to the number of its rows. To be more precise, if m_i is the degree of t_i then $\boldsymbol{A_{3i}}$ has $2m_i-4$ rows if t_i is boundary or $2m_i-3$ rows if t_i is singular or $2m_i-2$ rows otherwise.

Each matrix $\boldsymbol{A_{3i}}$ treats with a set of control points. Let $\mathcal{M}$ is the union of all such sets (for all matrices $\boldsymbol{A_{3i}},\ i = 1, \ldots, V$). We are going to create a new matrix $\boldsymbol{A^3}$ which has to be something like the union of all matrices $\boldsymbol{A_{31}}, \ldots, \boldsymbol{A_{3V}}$. First, let $\boldsymbol{A'_{3i}}$ is such matrix, that $\boldsymbol{A'_{3i}} \sim \boldsymbol{A_{3i}}$ and $\boldsymbol{A'_{3i}}$ has $|\mathcal{M}|$ columns, each one belongs to one of the points from $\mathcal{M}$. The order of the columns is the same in all matrices $\boldsymbol{A'_{31}}, \ldots, \boldsymbol{A'_{2V}}$ in this meaning: if $\mathcal{M} = \{b_1, b_2, \ldots, b_V\}$, then j-th column of $\boldsymbol{A'_{3i}}$ belongs to b_j; $j = 1, \ldots, |\mathcal{M}|,\ i = 1, \ldots, V$. Further, if $\boldsymbol{A_{3i}}$ treats with control point b_j, then j-th column of $\boldsymbol{A'_{3i}}$ equals to the corresponding column of $\boldsymbol{A_{3i}}$, otherwise the j-th column of $\boldsymbol{A'_{3i}}$ is zero.

Now, $\boldsymbol{A^3}$ is a block matrix with all matrices $\boldsymbol{A'_{3i}}$ as its elements:

$$\boldsymbol{A^3} = \begin{pmatrix} \boldsymbol{A'_{31}} \\ \boldsymbol{A'_{32}} \\ \cdots \\ \boldsymbol{A'_{3V}} \end{pmatrix}$$

Lemma 6.8 $\boldsymbol{A^3}$ *is regular for* $n \geq 4$.

Proof: If $n > 4$ then for each pair $\boldsymbol{A_{3i}}, \boldsymbol{A_{3j}}$; $i \neq j,\ i, j = 1, \ldots, V$ the sets of unknowns of their columns are disjunct, so the assertion of lemma for $n > 4$ follows imidiatelly. For $n = 4$ there are some pairs of matrices having a common unknown. For example on Figure 9, the matrix $\boldsymbol{A_{30}}$ of t_0 and the matrix $\boldsymbol{A_{3i}}$ belonging to $t_i,\ i = 1, \ldots, m$ have common unknown b^i_{20}. So $\boldsymbol{A_{30}}$ shares $b^1_{20}, \ldots, b^m_{20}$ with other matrices for vertices of the star. The rest unknowns are not shared.

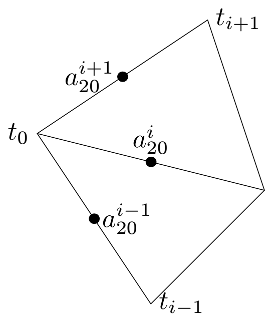

Figure 9: When considering $S_4^1(\mathcal{T})$, then the matrices of two neighbouring vertices t_0 and t_i share the control point over a_{20}^i.

Let's divide $\boldsymbol{A_{3i}}$ into the blocks similarly to $\boldsymbol{A_{2i}}$:

$$\boldsymbol{A_{3i}} = \begin{pmatrix} \boldsymbol{B'_{1i}} & \boldsymbol{B'_{2i}} & \boldsymbol{B'_{0i}} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A'_{1i}} \end{pmatrix}$$

Then columns of the matrix $\boldsymbol{B'_{2i}}$ belong to shared unknowns. Using the suitable order of the points in $\mathcal{M}$ we can gain, that the matrix $\boldsymbol{A^3}$ looks like follows:

$$\boldsymbol{A^3} \sim \begin{pmatrix} \boldsymbol{C_1} & \boldsymbol{O} & \dots & \boldsymbol{O} & \boldsymbol{R_1} \\ \boldsymbol{O} & \boldsymbol{C_2} & \dots & \boldsymbol{O} & \boldsymbol{R_2} \\ \dots & & & & \\ \boldsymbol{O} & \boldsymbol{O} & \dots & \boldsymbol{C_V} & \boldsymbol{R_V} \end{pmatrix}$$

where

$$\boldsymbol{C_i} = \begin{pmatrix} \boldsymbol{B'_{1i}} & \boldsymbol{B'_{0i}} \\ \boldsymbol{O} & \boldsymbol{A'_{1i}} \end{pmatrix}$$

where $\boldsymbol{O}$ is zero matrix of the needed type and $\boldsymbol{R_i}$ contains rest of columns of $\boldsymbol{A'_{3i}}$ not included in $\boldsymbol{C_i}$ and moreover the zero columns for all other shared unknowns in the matrix $\boldsymbol{A^3}$. Since $\boldsymbol{C_i}$ is diagonal for each $i = 1, \dots, V$, then also whole mentioned matrix is diagonal with no zero row and therefore $\boldsymbol{A^3}$ is regular. ■

The last matrix we are going to treat with in this section is

$$\boldsymbol{A'} = \begin{pmatrix} \boldsymbol{A^3} & \boldsymbol{O} \\ \boldsymbol{D_1} & \boldsymbol{D_2} \end{pmatrix},$$

where $\boldsymbol{D} = (\boldsymbol{D_1}\boldsymbol{D_2})$ is the matrix of all conditions (10) which are not included in the matrix $\boldsymbol{A^3}$. Figure 10 illustrates the situation. There is a quadruple of points which must satisfy (10) and this condition is not included in $\boldsymbol{A_{3i}}$ neither in $\boldsymbol{A_{3j}}$

Lemma 6.9 *$\boldsymbol{A} \sim \boldsymbol{A'}$ and $\boldsymbol{A'}$ is regular for $n \geq 4$.*

Proof: For $n = 4$ the matrix $\boldsymbol{D}$ is empty - with no row and $\boldsymbol{A'} = \boldsymbol{A^3}$. For $n > 4$ there are $(n-4)E_I$ rows in $\boldsymbol{D}$, E_I is number of interior edges.

The fact that $\boldsymbol{A} \sim \boldsymbol{A'}$ is obvious. Each row of $\boldsymbol{D}$ includes an unknown which occurs only once in all system of conditions determined by $\boldsymbol{A}$, i.e. for each such unknown there is only one non-zero value in its column. It deals with vertices marked on Figure 10 by big circle. Thanks to that fact and to the regularity of $(\boldsymbol{A^3}\boldsymbol{O})$ the whole matrix $\boldsymbol{A'}$ is regular. ■

Theorem 6.10

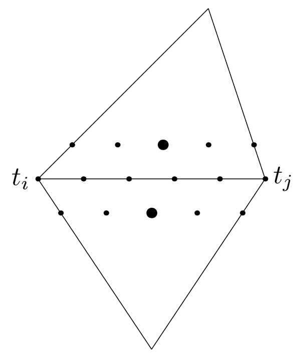

Figure 10: For a function from $S_5^1(\mathcal{T})$ and for each couple of neighbouring vertices there is a quadruple of points t_i, t_j which must satisfy a condition not included in the matrix for t_i neither in the matrix for t_j. Two points of the quadruple are denoted by big circle.

a) $r(\boldsymbol{A}) = (3n-2)V + (2-2n)V_b + 3n(N-2) - V_s$ *for* $n \geq 4$

b) $\dim(S_n^1(\mathcal{T})) = (n-1)(n-2)V - \frac{1}{2}(n-1)(n-4)V_b + (n^2-3n-1)(N-2) + V_s$ *for* $n \geq 4$

where V_s *is number of singular vertices of* $\mathcal{T}$.

Proof: From the Lemma 6.9 $r(\boldsymbol{A})$ equals number of rows of the matrix $\boldsymbol{A}'$. During the counting we will also use following assertions. They are very simple, the proofs of them are let to the reader:
1.) The sum of degrees of all vertices in Ω is $\sum_{i=1}^{V} \deg\ t_i = 2E$
2.) $E_I = E - V_b$

$$\begin{aligned}
r(\boldsymbol{A}) &= r(\boldsymbol{A}') = \sum_{i=1}^{V}(2\ deg\ t_i - 2) - (V_s + 2V_b) + (n-4)E_I = \\
&= 4E - 2V - V_s - 2V_b + (n-4)(E - V_b) = \\
&= nE - (n-2)V_b - 2V - V_s = \\
&= n(3V - V_b + 3(N-2)) - (n-2)V_b - 2V - V_s = \\
&= (3n-2)V + (2-2n)V_b + 3n(N-2) - V_s
\end{aligned}$$

For the number of the control points from (13)

$$\begin{aligned}
\#CP &= V + (n-1)E + \frac{(n-1)(n-2)}{2}F = \\
&= V + (n-1)(3V - V_b + 3(N-2)) + \frac{1}{2}(n-1)(n-2)(2V - V_b + 2(N-2)) = \\
&= n^2V + \frac{1}{2}(n-n^2)V_b + (n^2-1)(N-2)
\end{aligned}$$

And so finally we can compute the dimension of the space $S_n^1(\mathcal{T})$:

$$\begin{aligned}
\dim(S_n^1(\mathcal{T})) &= \#CP - r(\boldsymbol{A}) = \\
&= n^2V + \frac{1}{2}(n-n^2)V_b + (n^2-1)(N-2) - \\
&\qquad ((3n-2)V + (2-2n)V_b + 3n(N-2) - V_s) = \\
&= (n-1)(n-2)V - \frac{1}{2}(n-1)(n-4)V_b + (n^2-3n-1)(N-2) + V_s
\end{aligned}$$

■

7 Basis of $S_4^1(\mathcal{T})$

7.1 Ceva Theorem

To introduce the famous Ceva theorem we need to recall the notion of ratio of three collinear points. Let A, B, C be three collinear points such, that $C \neq A$ and $C \neq B$. Then formally we define (ABC) – the desired ratio – as

$$(ABC) = \frac{C - A}{C - B}$$

Theorem 7.1 (Ceva) *Let A, B, C be three non-collinear points and A', B', C' be three points on lines BC, CA, AB respectively and $\{A, B, C\} \cap \{A', B', C'\} = \emptyset$. Then lines AA', BB' and CC' meet in common point iff $(ABC')(BCA')(CAB') = -1$.*

The proof is elementary and can be found in many books concerning plane geometry or affine spaces.

We would like to express Ceva theorem in barycentric calculus. Let $\mathrm{d}_{or}(ABC)$ be the oriented distance of point A to line BC and $\mathrm{d}_{or}(A, B, C) > 0$ iff $\triangle ABC$ is oriented counterclockwise. Let the points be assigned according to Figure 11. Let $\tilde{t}_i = (\lambda_{i0}, \lambda_{i1}, \lambda_{i2})$, $i = 0, 1, 2$ be point in barycentric coordinates w.r.t. $\triangle t_0 t_1 t_2$. Then we can write for $\tilde{t}_i$

$$\begin{aligned} \frac{\lambda_{i,i+1}}{\lambda_{i,i-1}} &= \frac{\det(\tilde{t}_i t_{i-1} t_i)}{\det(t_{i+1} \tilde{t}_i t_i)} = \frac{\mathrm{d}_{or}(\tilde{t}_i t_{i-1} t_i)}{\mathrm{d}_{or}(t_{i+1} \tilde{t}_i t_i)} = \\ &= -\frac{(\bar{t}_i - t_{i-1})}{(\bar{t}_i - t_{i+1})} = -(t_{i-1} t_{i+1} \bar{t}_i) \end{aligned}$$

Immediately we can write

$$\frac{\lambda_{0,1}}{\lambda_{0,2}} \frac{\lambda_{1,2}}{\lambda_{1,0}} \frac{\lambda_{2,0}}{\lambda_{2,1}} = (-1)^3.(t_1 t_2 \bar{t}_0)(t_2 t_0 \bar{t}_1)(t_0 t_1 \bar{t}_2)$$

Thus in Ceva theorem can be the ratio condition rewritten as

$$\frac{\lambda_{0,1}}{\lambda_{0,2}} \frac{\lambda_{1,2}}{\lambda_{1,0}} \frac{\lambda_{2,0}}{\lambda_{2,1}} = 1 \tag{17}$$

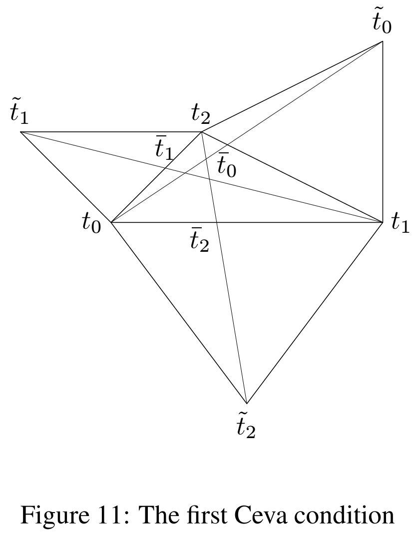

Figure 11: The first Ceva condition

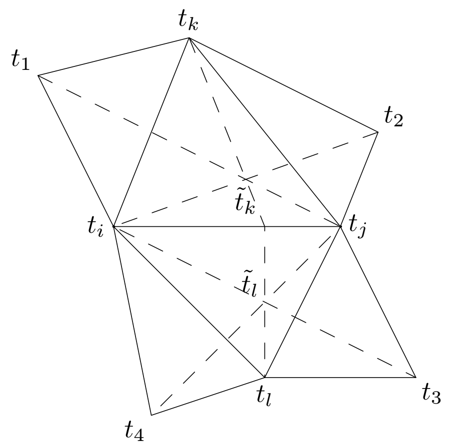

Figure 12: The second Ceva condition

Definition 7.2 *We will say that $\triangle t_0t_1t_2 \in \mathcal{T}$ satisfies the* ***first Ceva condition*** *iff there exist 3 neighbouring triangles $\triangle \tilde{t}_0t_1t_2$, $\triangle t_0\tilde{t}_1t_2$, $\triangle t_0t_1\tilde{t}_2 \in \mathcal{T}$ and the lines $t_0\tilde{t}_0$, $t_1\tilde{t}_2$, $t_2\tilde{t}_2$ coincide in common point. It is equivalent to algebraic condition (17).*

Definition 7.3 *We will say, that two neighbouring triangles $\triangle t_it_jt_k$, $\triangle t_it_jt_l \in \mathcal{T}$ satisfy the* ***second Ceva condition*** *iff there exist 4 neighbouring triangles $\triangle t_it_kt_1$, $\triangle t_jt_kt_2$, $\triangle t_jt_lt_3$, $\triangle t_it_lt_4 \in \mathcal{T}$ (see Figure 12) for which the following holds: if $\tilde{t}_k$ is the intersection of t_it_2, t_jt_1 and $\tilde{t}_l$ is the intersection of t_it_3, t_jt_4, then lines $t_k\tilde{t}_k$, $t_l\tilde{t}_l$, t_it_j coincide in common point.*

The second Ceva condition has also its mathematic expression in barycentric calculus. It is presented in the following theorem.

Theorem 7.4 *Let $\triangle t_it_jt_k$, $\triangle t_it_jt_l$, t_1, t_2, t_3, t_4 be as in Definition 7.3. Let $(\lambda_{m0}, \lambda_{m1}, \lambda_{m2})$, $m = 1, 2$ are the barycentric coordinates of t_m w.r.t. $\triangle t_it_jt_k$ and $(\lambda_{m0}, \lambda_{m1}, \lambda_{m2})$, $m = 3, 4$ are barycentric coordinates of t_m w.r.t. $\triangle t_it_jt_l$. Then $\triangle t_it_jt_k$, $\triangle t_it_jt_l$ satisfy the second Ceva condition iff*

$$\frac{\lambda_{12}}{\lambda_{10}}\frac{\lambda_{21}}{\lambda_{22}} = \frac{\lambda_{42}}{\lambda_{40}}\frac{\lambda_{31}}{\lambda_{32}} \tag{18}$$

Proof: Let (18) holds. Let $\bar{t}_k, \bar{t}_l \in t_it_j$ are such points, that lines t_it_2, t_jt_1, $t_k\bar{t}_k$ coincide in a common point $\tilde{t}_k$ and similarly t_it_3, t_jt_4, $t_l\bar{t}_l$ coincide in $\tilde{t}_l$. Let $\bar{t}_k = \lambda_{k0}t_i + \lambda_{k1}t_j$ and $\bar{t}_l = \lambda_{l0}t_i + \lambda_{l1}t_j$. Then from (17)

$$\frac{\lambda_{12}}{\lambda_{10}}\frac{\lambda_{21}}{\lambda_{22}} = \frac{\lambda_{k1}}{\lambda_{k0}} \quad \text{and} \quad \frac{\lambda_{42}}{\lambda_{40}}\frac{\lambda_{31}}{\lambda_{32}} = \frac{\lambda_{l1}}{\lambda_{l0}}.$$

Then from (18)

$$\frac{\lambda_{k1}}{\lambda_{k0}} = \frac{\lambda_{l1}}{\lambda_{l0}}$$

Since $\lambda_{k0} + \lambda_{k1} = \lambda_{l0} + \lambda_{l1} = 1$, we have $\bar{t}_k = \bar{t}_l$ and therefore $\triangle t_it_jt_k$, $\triangle t_it_jt_l$ satisfy the second Ceva condition.

Now, let the second Ceva condition is satisfied. Let $t \in t_it_j$, $t = \lambda_0t_i + \lambda_1t_j$ is such point that

$$\frac{\lambda_{12}}{\lambda_{10}}\frac{\lambda_{21}}{\lambda_{22}}\frac{\lambda_0}{\lambda_1} = 1 \text{ and } \frac{\lambda_{42}}{\lambda_{40}}\frac{\lambda_{31}}{\lambda_{32}}\frac{\lambda_0}{\lambda_1} = 1$$

and it directly implies (18). ■

7.2 Constraints

As it was mentioned, we will focus on $S_4^1(\mathcal{T})$. We require that the triangulation $\mathcal{T}$ satisfies three following conditions:

1. Let $[t_i, t_j]$ is an interior edge and the triangles $\triangle t_i t_j t_k \in \mathcal{T}$ and $\triangle t_i t_j t_l \in \mathcal{T}$. We require that the points t_k, t_i, t_l are non-collinear and also the points t_k, t_j, t_l are non-collinear. (see Figure 13)
2. No triangle from $\mathcal{T}$ satisfies the first Ceva condition.
3. No two neighbouring triangles satisfy the second Ceva condition.

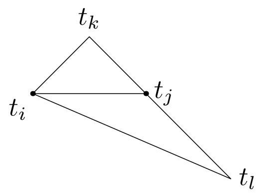

Figure 13: Unacceptable situation of two neighbouring triangles (condition 1)

Note 7.5 *Notice, that the first condition implies, that there is no singular point in the triangulation $\mathcal{T}$.*

Note 7.6 *Let us have a brief look at barycentric coordinates in $\mathcal{T}$. Let $\triangle t_0 t_1 t_2, \triangle \tilde{t}_0 t_1 t_2 \in \mathcal{T}$ have a common edge $[t_1 t_2]$. Let $\lambda_0, \lambda_1, \lambda_2$ are barycentric coordinates of $\tilde{t}_0$ w.r.t. $\triangle t_0 t_1 t_2$:*

$$\tilde{t}_0 = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$$

Then

$$\lambda_0 \neq 0, \quad \lambda_1 \neq 0, \quad \lambda_2 \neq 0.$$

Really, if $\lambda_0 = 0$ then $\tilde{t}_0$ lies on the line determined by $[t_1 t_2]$ and $\triangle \tilde{t}_0 t_1 t_2$ is not a proper triangle. If $\lambda_1 = 0$, then $\tilde{t}_0, t_1, t_2$ are collinear and the situation does not respect the first constraint put on the triangulation. The case $\lambda_2 = 0$ is similar to $\lambda_1 = 0$.

According to Theorem 6.10 and Note 7.5 it is very simple to convince ourselves that the dimension of $S_4^1(\mathcal{T})$ satisfies

$$\dim(S_4^1(\mathcal{T})) = 6V + 3(N - 2) \tag{19}$$

Definition 7.7 *Let $dim = \dim(S_4^1(\mathcal{T}))$. Let $E_i : \Omega \to R;\ i = 1, \ldots, dim$ is a basis of $S_4^1(\mathcal{T})$. Each $F \in S_4^1(\mathcal{T})$ can be uniquely expressed as a linear combination of the basis elements:*

$$F = d_1 E_1 + d_2 E_2 + \ldots + d_{dim} E_{dim}.$$

*Points $d_i \in R;\ i = 1, \ldots, dim$ are called **control points** of F w.r.t. the basis $E_1, \ldots, E_{dim}$.*

For each E_j from the basis of $S_4^1(\mathcal{T})$ we will try to reach as small support as possible. Smaller support of E_j causes "more local" alteration of the function $F = \sum d_i E_i$ when the control point d_j is moved.

The second required condition is that the support of each basis function E_j has just one boundary component (is simply connected).

Now, we will start looking for the basis of $S_4^1(\mathcal{T})$. We will proceed as follows:

First we will ask, if Ω_j is a suitable support for a basis function E_j. It has to be large enough to be possible to define E_j over it. For instance, E_j can not be defined such that its support consists of the only triangle which all 3 vertices are interior w.r.t. Ω, because the $E_j(u) = 0\ \forall u \in \Omega$ and so it is not basis function.

In the second step, when the suitable support Ω_j for a basis function is found, we will define E_j such that $\operatorname{supp} E_j = \Omega_j$.

7.3 Basic Basis Functions

In this section we formulate the construction of basis functions.

Let $\Omega_j \subset \Omega$ and $\mathcal{T}_j \subset \mathcal{T}$ is the corresponding triangulation of Ω_j. Then $V^j, V_B^j, V_I^j, \ldots$ denote the number of vertices from $\mathcal{T}_j$, boundary (w.r.t. $\mathcal{T}_j$) vertices from $\mathcal{T}_j$, interior vertices from $\mathcal{T}_j$ and so on. Let Ω_j has just one boundary.

Further let $S_4^1(\mathcal{T}_j)$ is defined similarly to $S_4^1(\mathcal{T})$:

$$S_4^1(\mathcal{T}_j) := \{F \in C^1(\Omega_j); F|_\triangle \in P_n\ \forall \triangle \in \mathcal{T}_j\}$$

We will treat with a special subspace of $S_4^1(\mathcal{T}_j)$.

Definition 7.8 *Let $\mathcal{T}_j$ is the triangulation of Ω_j. Then let $\partial\Omega_j$ be the boundary of Ω_j. $\tilde{S}_4^1(\mathcal{T}_j)$ is such subspace of $S_4^1(\mathcal{T})$ that the following conditions hold for each $F \in \tilde{S}_4^1(\mathcal{T}_j)$ and $u \in \partial\Omega_j$:*

$$\begin{aligned} F(u) &= 0 \\ D_e F(u) &= 0 \quad \textit{for each vector } e, \end{aligned}$$

where $D_e F$ is the directional derivative of F with direction vector e.

Lemma 7.9 *Let $\mathcal{T}_j \subset \mathcal{T}$ is the triangulation of $\Omega_j \subset \Omega$. If $\dim(\tilde{S}_4^1(\mathcal{T}_j)) > 0$ then there exists a non-zero function $E_j \in S_4^1(\mathcal{T})$ such that $\operatorname{supp} E_j = \Omega_j$ and moreover $E_j|_{\Omega_j} \in \tilde{S}_4^1(\mathcal{T}_j)$.*

Proof: Let $\dim(\tilde{S}_4^1(\mathcal{T}_j)) > 0$. Then there exists a non-zero $E'_j \in \tilde{S}_4^1(\mathcal{T}_j)$ Let us define E_j as follows:

$$E_j(u) = \begin{cases} E'_j(u) & \text{if } u \in \Omega_j \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $u \in \partial\Omega_j$ holds $E'_j(u) = 0$ and $D_e E'_j(u) = 0$ for each e, the function E_j is C^1 continuous and thus $E_j \in S_4^1(\mathcal{T})$. The properties $\operatorname{supp} E_j = \Omega_j$ and $E_j|_{\Omega_j} = E'_j$ follow from the definition of E_j. ■

Note 7.10 *The reverse implication of the previous lemma does not hold if $\partial\Omega_j \cap \partial\Omega \neq \emptyset$. In this case all functions from $\tilde{S}_4^1(\mathcal{T}_j)$ has to be zero on the common part of the boundary, even all its directional derivatives have to be zero here. But for the functions from $S_4^1(\mathcal{T})$ it is not required.*

To avoid complicated situations occurring on the boundary of Ω, first we will restrict ourselves to the domains with no boundary. Boundary conditions will be added later.

Definition 7.11 *Let t be the vertex from $\mathcal{T}$. Then **the star of the vertex t** is the subspace $\Omega(t) \subset \Omega$, which is defined as follows:*

$$\begin{aligned} \mathcal{T}(t) = \{t\} \cup \{t_i;\ [t, t_i] \in \mathcal{T}\} \cup \{[t, t_i];\ [t, t_i] \in \mathcal{T}\} \cup \\ \{[t_i, t_j];\ [t, t_i] \in \mathcal{T}, [t, t_j] \in \mathcal{T}\} \cup \{\triangle t t_i t_j;\ \triangle t t_i t_j \in \mathcal{T}\} \end{aligned}$$

is the triangulation of the star of t and

$$\Omega(t) = \bigcup P; P \in \mathcal{T}(t).$$

Lemma 7.12 *Let Ω_j is the star of the interior vertex $t_j \in \mathcal{T}$. Then*

$$\dim(\tilde{S}_4^1(\mathcal{T}_j)) = 3.$$

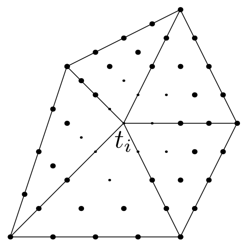

Figure 14: The support of $F \in \tilde{S}_4^1(\mathcal{T}_j)$. The zero Bézier control point are sketched as black circles.

Proof: According to (19)

$$\dim(S_4^1(\mathcal{T}_j)) = 6V^j - 3 = 6(1 + m_j) = 6m_j + 3,$$

where m_j is $\deg t_j$. The dimension of $\tilde{S}_4^1(\mathcal{T}_j)$ is smaller, because of the constraints put on the boundary of $\tilde{S}_4^1(\mathcal{T}_j)$.

For $F \in \tilde{S}_4^1(\mathcal{T}_j)$ holds $F(u) = 0;\ u \in \partial\Omega_j$. It implies that all Bézier control points of F defined over the boundary of Ω_j are zero. Moreover, since $D_e F(u) = 0$ for each $u \in \partial\Omega_j$, also the second line of control points neighbouring to the boundary, contains only zero Bézier control points (see Figure 14). The dimension of $\tilde{S}_4^1(\mathcal{T}_j)$ can be expressed as a number of Bézier control points which values can be arbitrarily chosen if the mentioned points are zero.

To construct a function from $S_4^1(\mathcal{T}_j)$ we need to set

$$\dim(S_4^1(\mathcal{T}_j)) = 6m_j + 3$$

suitable values of its Bézier control points, the others are determined by C^1 continuity. But if $F \in \tilde{S}_4^1(\mathcal{T}_j)$, the properties of this subspace have chosen several values of the to be zero. Let us count them.

For each vertex from the boundary of Ω_j let us consider 4 Bézier control points: one defined exactly over it and 3 defined over the neighbouring abscissae lying in the incident edges from $\mathcal{T}_j$. Let us choose 3 non-collinear of them and set them to zero. Then because of the C^1 continuity, also the 4-th is zero. We have set $3m_j$ control points.

For each boundary edge from $\mathcal{T}_j$ (boundary w.r.t. Ω_j) let us set the rest control point defined over it to zero. We have set m_j control points.

For each boundary edge from $\mathcal{T}_j$ there are 2 Bézier control points of the neighbouring line, which has to be set to zero. Doing it we have set $2m_j$ control points.

After counting them, we see, that we have set $6m_j$ Bézier control points. So for each $F \in \tilde{S}_4^1(\mathcal{T}_j)$ there are only $\dim(S_4^1(\mathcal{T}_j)) - 6m_j = 3$ control points left to be arbitrarily set and thus $\dim(\tilde{S}_4^1(\mathcal{T}_j)) = 3$. ■

Thanks to Lemma 7.12 we know, that the star of the interior vertex is large enough to construct up to 3 linearly independent functions such that Ω_j is their support.

Let $[t_i t_j]$ is an edge from $\mathcal{T}$, $E \in \mathcal{B}_1$. Denote $a_0, \ldots, a_4$ abscissae on $[t_i t_j]$. Let $b_0, \ldots, b_4$ are corresponding Bézier control points. If E is a function from the basis of Ω_i, then $b_2 = b_3 = b_4 = 0$. If E is a function from the basis of Ω_j, then $b_0 = b_1 = b_2 = 0$. Otherwise $b_0 = b_1 = \ldots = b_4 = 0$. We see, that $b_2 = 0$ for each $E \in \mathcal{B}_1$. So, we are looking for such functions, which have non-zero control point assigned as b_2.

Definition 7.13 *Let $[t_i, t_j]$ be the edge of $\mathcal{T}$. Then* ***the star of the edge*** *$[t_i, t_j]$ is the subspace $\Omega([t_i, t_j]) \subset \Omega$ which is defined as follows:*

$$
\begin{aligned}
\mathcal{T}([t_i, t_j]) = \{&t_i, t_j, [t_i, t_j]\} \cup \\
&\{t_k, [t_i, t_k], [t_j, t_k], \triangle t_i t_j t_k;\ \forall t_k : \triangle t_i t_j t_k \in \mathcal{T}\} \cup \\
&\{t_l, [t_i, t_l], [t_k, t_l], \triangle t_i t_k t_l;\ \forall t_k \forall t_l : \triangle t_i t_j t_k \in \mathcal{T}, \triangle t_i t_k t_l \in \mathcal{T}\} \cup \\
&\{t_m, [k_j, t_m], [t_k, t_m], \triangle t_j t_k t_m;\ \forall t_k \forall t_m : \triangle t_i t_j t_k \in \mathcal{T}, \triangle t_j t_k t_m \in \mathcal{T}\}, \\
\Omega([t_i, t_j]) = \bigcup &P; P \in \mathcal{T}([t_i, t_j]).
\end{aligned}
$$

Let t_i, t_j are interior vertices. Note, if $\deg t_i \geq 5$ and $\deg t_j \geq 5$ then the star of $[t_i, t_j]$ contains no interior vertex. But when $\deg t_i$ or $\deg t_j$ or both of them are at most 4, then the star of $[t_i, t_j]$ may contain one or two interior vertices (see Figure 15).

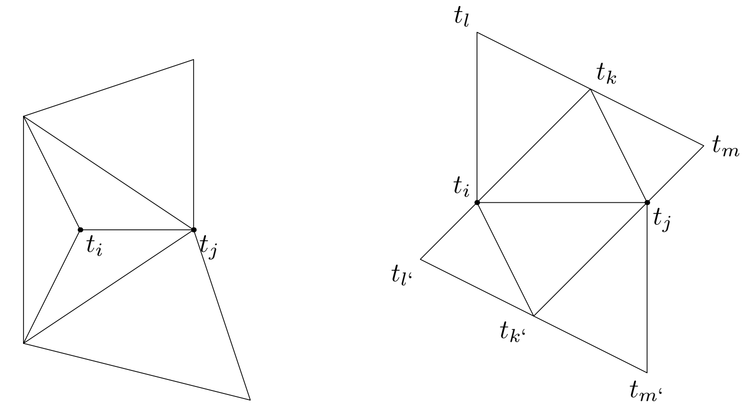

Figure 15: The star of the edge $[t_i t_j]$ – two different shapes: left one contains an interior vertex and the right one does not.

Lemma 7.14 *Let t_i, t_j are the interior vertices, $[t_i, t_j] \in \mathcal{T}$. Let $\Omega_{ij} \subset \Omega$ is the star of $[t_i, t_j]$. Then $\dim(\tilde{S}_4^1(\mathcal{T}_{ij})) \geq 1$ and, moreover, there exists $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ such that its Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: Let abscissae of $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ are denoted as in the Figure 16. Let b_i is Bézier control point over a_i for $i = 1, \ldots, 11$. Because of the conditions put on $\tilde{S}_4^1(\mathcal{T}_{ij})$, only $b_1, \ldots, b_{11}$ can have non-zero value, other control points are zero. Because of C^1 continuity there are just 10 conditions put on $b_1, \ldots, b_{11}$ (they are sketched on the Figure 16). It holds that $\dim(\tilde{S}_4^1(\mathcal{T}_{ij}))$ equals to the dimension of the solution space of the mentioned system of 10 equations and therefore $\dim(\tilde{S}_4^1(\mathcal{T}_{ij})) \geq 1$ We need only to find out if $\exists F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ such that its Bézier control point b_{11} is non-zero.

First we need to assign the barycentric coordinates of several points:

$$
\begin{aligned}
t_1 &= (\lambda_{10}, \lambda_{11}, \lambda_{12}) \text{ w.r.t. } \triangle t_i t_j t_0, \\
t_2 &= (\lambda_{20}, \lambda_{21}, \lambda_{22}) \text{ w.r.t. } \triangle t_i t_j t_0, \\
t_3 &= (\lambda_{30}, \lambda_{31}, \lambda_{32}) \text{ w.r.t. } \triangle t_i t_j t_5, \\
t_4 &= (\lambda_{40}, \lambda_{41}, \lambda_{42}) \text{ w.r.t. } \triangle t_i t_j t_5, \\
t_5 &= (\lambda_{50}, \lambda_{51}, \lambda_{52}) \text{ w.r.t. } \triangle t_i t_j t_0.
\end{aligned}
$$

The solution space S of the system of 10 equations can be expressed as an intersection $S = S_1 \cap S_2 \cap S_3$, where

S_1 is a solution space of 4 equations put on $b_1, \ldots, b_5$,
S_2 is a solution space of 4 equations put on $b_6, \ldots, b_{10}$,
S_3 is a solution space of 2 equations treating with b_{11}.

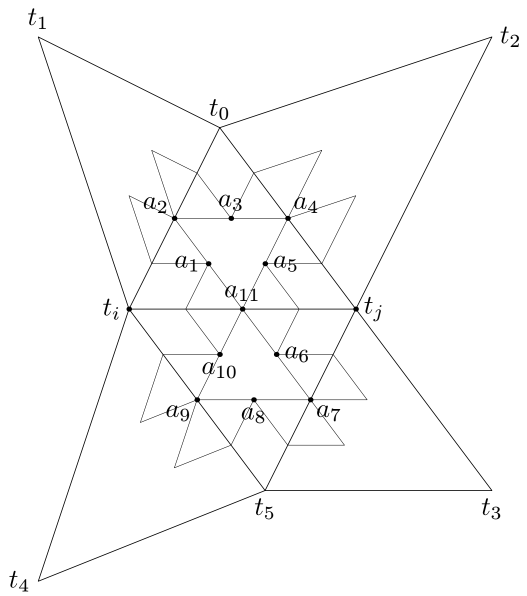

Figure 16: The star of the edge $[t_i t_j]$; a_i is abscissa of Bézier control point b_i.

Let us find S_1:

$$\begin{array}{rcl}
\lambda_{11}b_1+\lambda_{12}b_2 & = & 0 \\
\lambda_{10}b_2+\lambda_{11}b_3 & = & 0 \\
\lambda_{20}b_3+\lambda_{21}b_4 & = & 0 \\
\lambda_{22}b_4+\lambda_{20}b_5 & = & 0
\end{array}$$

Let b_3 is chosen arbitrarily. Then

$$\begin{array}{ll}
b_2 = -\frac{\lambda_{11}}{\lambda_{10}}b_3 & b_1 = \frac{\lambda_{12}}{\lambda_{10}}b_3 \\
b_4 = -\frac{\lambda_{20}}{\lambda_{21}}b_3 & b_5 = \frac{\lambda_{22}}{\lambda_{21}}b_3
\end{array}$$

All coefficients are non-zero (see Note 7.6 after the constraints put on the triangulation) and therefore they can occur in the denominator.

Let us find S_2:

$$\begin{array}{rcl}
\lambda_{30}b_6+\lambda_{32}b_7 & = & 0 \\
\lambda_{31}b_7+\lambda_{30}b_8 & = & 0 \\
\lambda_{41}b_8+\lambda_{40}b_9 & = & 0 \\
\lambda_{42}b_9+\lambda_{41}b_{10} & = & 0
\end{array}$$

Let b_8 is chosen arbitrarily. Then

$$\begin{array}{ll}
b_7 = -\frac{\lambda_{30}}{\lambda_{31}}b_8 & b_6 = \frac{\lambda_{32}}{\lambda_{31}}b_8 \\
b_9 = -\frac{\lambda_{41}}{\lambda_{40}}b_8 & b_{10} = \frac{\lambda_{42}}{\lambda_{40}}b_8
\end{array}$$

Let us find S_3 and then $S_1 \cap S_2 \cap S_3$:

$$\begin{array}{rcl}
b_{10} & = & \lambda_{52}b_1 + \lambda_{51}b_{11} \\
b_6 & = & \lambda_{52}b_5 + \lambda_{50}b_{11}
\end{array}$$

$$\begin{array}{rcl}
\dfrac{\lambda_{42}}{\lambda_{40}}b_8 & = & \lambda_{52}\dfrac{\lambda_{12}}{\lambda_{10}}b_3 + \lambda_{51}b_{11} \\
\dfrac{\lambda_{32}}{\lambda_{31}}b_8 & = & \lambda_{52}\dfrac{\lambda_{22}}{\lambda_{21}}b_3 + \lambda_{50}b_{11}
\end{array}$$

$$(\lambda_{50}\frac{\lambda_{31}}{\lambda_{32}} - \lambda_{51}\frac{\lambda_{40}}{\lambda_{42}})b_{11} \quad = \quad \lambda_{52}(\frac{\lambda_{12}\lambda_{40}}{\lambda_{10}\lambda_{42}} - \frac{\lambda_{22}\lambda_{31}}{\lambda_{21}\lambda_{32}})b_3$$

Since

$$(\lambda_{50}\frac{\lambda_{31}}{\lambda_{32}} \neq \lambda_{51}\frac{\lambda_{40}}{\lambda_{42}})b_{11}$$

(see the second constraint), a solution with non-zero b_{11} exists iff

$$\frac{\lambda_{12}\lambda_{40}}{\lambda_{10}\lambda_{42}} \neq \frac{\lambda_{22}\lambda_{31}}{\lambda_{21}\lambda_{32}} \tag{20}$$

This condition is always satisfied according to our assumption (the third constraint). ■

The proof of Lemma 7.14 is stronger than its assertion. It also implies the following properties:

Corollary 7.15 *Let F is the function satisfying the conditions of Lemma 7.14 and $b_1, \ldots, b_{11}$ are its Bézier control points (assigned as in the proof). Then*

1. $b_i \neq 0;\ i = 1, \ldots, 11,$
2. *The values of $b_1, \ldots, b_{11}$ can be obtained by setting the value of an arbitrary point of them and calculating all the others according to C^1 continuity conditions.*

7.4 Boundary Basis Functions

As noted after Lemma 7.9, to find a basis function of $S_4^1(\mathcal{T})$ which is non-zero somewhere on the boundary of Ω, we must proceed a bit differently.

Definition 7.16 *Let $\mathcal{T}$ is the triangulation of the given domain Ω. Then **the extended triangulation $\mathcal{T}'$** is defined as follows:*

$$\begin{aligned}
\mathcal{T}_1 \;&=\; \mathcal{T} \cup \{t_{ij}, [t_i, t_{ij}], [t_j, t_{ij}] \triangle t_i t_j t_{ij};\\
&\qquad t_{ij} \textit{ is exterior to } \Omega \textit{ and not at } t_i t_j \textit{ line};\ [t_i, t_j] \in \mathcal{T} \textit{ is boundary edge w.r.t. } \mathcal{T}\}\\
\mathcal{T}' \;&=\; \mathcal{T}_1 \cup \{[t_{ij}, t_{jk}], \triangle t_j t_{ij} t_{jk};\\
&\qquad t_j \in \mathcal{T} \textit{ is boundary vertex w.r.t. } \mathcal{T} \textit{ and } \deg t_j \geq 5 \textit{ w.r.t. } \mathcal{T}_1\} \cup\\
&\quad \{t_{j0}, [t_j, t_{j0}], [t_{ij}, t_{j0}], [t_{j0}, t_{jk}], \triangle t_j t_{ij} t_{j0}, \triangle t_j t_{j0} t_{jk};\\
&\qquad t_j \in \mathcal{T} \textit{ is boundary w.r.t. } \mathcal{T} \textit{ and } \deg t_j = 4 \textit{ w.r.t. } \mathcal{T}_1\}
\end{aligned}$$

*All added objects are non-degenerated and the objects of the same dimension have non-intersecting relative interiors. The constraints put on the triangulation $\mathcal{T}$ are valid also for $\mathcal{T}'$. Then $\Omega' = \bigcup P; P \in \mathcal{T}'$ is called **the extended domain**.*

Definition 7.17 *Let Ω_j is the star of the boundary vertex $t_j \in \Omega$. Let Ω'_j is the star of the vertex t_j in the extended domain Ω', let $\mathcal{T}'_j$ is the corresponding triangulation of Ω'_j. Then*

$$\bar{S}_4^1(\mathcal{T}_j) = \{F \in S_4^1(\mathcal{T}_j); F = F'|_\Omega \textit{ where } F' \in \tilde{S}_4^1(\mathcal{T}'_j)\}.$$

Lemma 7.18 *Let $\Omega_j \subset \Omega$ is the star of the boundary vertex $t_j \in \mathcal{T}$. Then $\dim(\bar{S}_4^1(\mathcal{T}_j)) = 4$.*

Proof: It is very similar to the proof of Lemma 7.12. ■

Definition 7.19 *Let Ω_{ij} is the star of the edge $[t_i, t_j] \in \Omega$, where t_i or t_j or both of them are boundary. Let Ω'_{ij} is the star of the edge $[t_i, t_j]$ in the extended domain Ω'. Then*

$$\bar{S}_4^1(\mathcal{T}_{ij}) = \{F \in S_4^1(\mathcal{T}_{ij}); F = F'|_\Omega \textit{ where } F' \in \tilde{S}_4^1(\mathcal{T}'_{ij})\}.$$

Lemma 7.20 *Let $\Omega_{ij} \subset \Omega$ is the star of $[t_i, t_j] \in \mathcal{T}$, where t_i or t_j or both of them are the boundary vertices (w.r.t. $\mathcal{T}$). Then $\dim(\tilde{S}_4^1(\mathcal{T}_{ij})) \geq 1$ and moreover there exists $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ such that its Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: This lemma is direct corollary of Lemma 7.14. ■

7.5 Independence Of Functions

This section deals with the independence of above constructed functions. Actually, we prove, that they form a basis of $S_4^1(\mathcal{T})$.

Definition 7.21 *Let Ω_j is the star of $t_j \in \mathcal{T}$. If t_j is the interior (boundary) vertex, then the triplet (quadruple) of linearly independent functions $E_{ji} \in S_4^1(\mathcal{T})$; $i = 1, \ldots, 3(4)$ such that*

1. supp $E_{ji} = \Omega_j$
2. $E_{ji}|_{\Omega_j} \in \tilde{S}_4^1(\mathcal{T}_j)$ $(\bar{S}_4^1(\mathcal{T}_j))$

*is called **basis of the star of t_j**.*

Let $\mathcal{B}_1 \subset S_4^1(\mathcal{T})$, $\mathcal{B}_1$ contains $3V_I + 4V_B = 3V + V_B$ functions (for each vertex t_j there is the basis of its star).

Lemma 7.22 *$\mathcal{B}_1$ is linearly independent set of functions.*

Proof: The functions from $\mathcal{B}_1$ are linearly independent iff their control nets are linearly independent. The second assertion is true (the considerations are very similar to the ones we will do more detailed in the proof of Lemma 7.25. ■

Definition 7.23 *Let $t_i, t_j \in \mathcal{T}$, $[t_i, t_j] \in \mathcal{T}$ and let Ω_{ij} is the star of $[t_i, t_j]$. Let $E_{ij} \in S_4^1(\mathcal{T})$ is such function that*

1. supp $E_{ij} = \Omega_{ij}$
2. $E_{ij}|_{\Omega_{ij}} \in \tilde{S}_4^1(\mathcal{T}_{ij})(\bar{S}_4^1(\mathcal{T}_{ij})$
3. *Bézier control point over the medium of $[t_i, t_j]$ is non-zero.*
4. $\forall t \in \mathcal{T} : E_{ij}(t) = 0$ *and* $D_e E_{ij}(t) = 0$ *for each* e.

*Then E_{ij} is called **basis function over the edge** $[t_i, t_j]$.*

Note 7.24 *Definition of the basis function over the edge is correct. For each edge there exists such function (according to Lemma 7.14 and Lemma 7.20 it is always possible to find E'_{ij} satisfying first three conditions from the definition).*

If $\deg t_i \geq 5$ and $\deg t_j \geq 5$ than supp E'_{ij} has no interior edge and thus $E_{ij} = E'_{ij}$ is the basis function over $[t_i, t_j]$.

If $\deg t_i \leq 4$ then t_i is the interior vertex of supp E'_{ij}. Let E_{i0}, E_{i1}, E_{i2} is a basis of the star of t_i. Let $E_i = d_0 E_{i0} + d_1 E_{i1} + d_2 E_{i2}$ is such function that $E_i(t_i) = E'_{ij}(t_i)$ and $D_e E_i(t_i) = D_e E'_{ij}(t_i)$ $\forall e$. According to the definition of the basis of the star it is always possible to find such function. Then $E_{ij} = E'_{ij} - E_i$ is the desired basis function over the edge $[t_i, t_j]$.

The case of $\deg t_j \leq 4$ is similar.

Let $\mathcal{B}_2$ is such subset of $S_4^1(\mathcal{T})$ that for each edge $[t_i, t_j]$ there is one of basis functions over $[t_i, t_j]$ in $\mathcal{B}_2$.

Lemma 7.25 *$\mathcal{B}_2$ is linearly independent set of functions.*

Proof: If $\mathcal{B}_2$ is not linearly independent set, then $\exists F_i \in \mathcal{B}_2$ such that it is possible to omit it, i.e. the space generated by $\mathcal{B}_2$ and the space generated by $\mathcal{B}_2 \setminus \{F_i\}$ are equal. We will show that no function can be omitted and so $\mathcal{B}_2$ is linearly independent set.

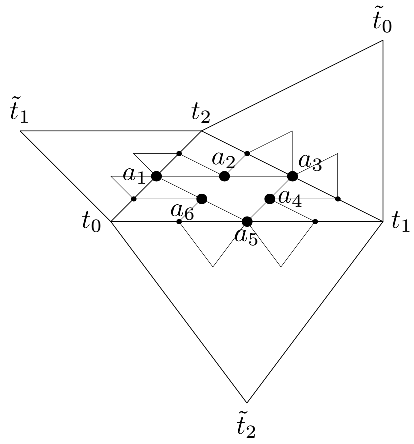

Figure 17: Triangle $t_0t_1t_2$ and its neighbours; a_i is abscissa of Bézier control point b_i.

Let $\triangle t_0t_1t_2 \in \mathcal{T}$. We will treat with $\triangle t_0t_1t_2$ and the neighbouring triangles. The whole situation is illustrated in Figure 17. Let F_0 is the basis function over the edge $[t_1, t_2]$, F_1 over $[t_2, t_1]$ and F_2 over $[t_0, t_1]$. Further, let $\tilde{t}_i = \lambda_{i0}t_0 + \lambda_{i1}t_1 + \lambda_{i2}t_2;\ i = 0, 1, 2$.

Let $F = \sum d_kF_k;\ F_k \in \mathcal{B}_2$. The same equations hold for control nets of F, F_k.

Let b_1, b_3, b_5 are Bézier control points of F over abscissae a_1, a_3, a_5 (see Figure 17). Let b_1^i, b_3^i, b_5^i are Bézier control points of F_i over abscissae a_1, a_3, a_5, $i = 0, 1, 2$. The control points b_1, b_3, b_5 of F are affected only by corresponding points of F_0, F_1 and F_2, i.e.

$$b_j = d_0b_j^0 + d_1b_j^1 + d_2b_j^2; \quad j = 1, 3, 5.$$

We will prove, that for each triplet of values b_1, b_3, b_5 $\exists F; F = d_0F_0 + d_1F_1 + d_2F_2$ such that b_1, b_3, b_5 are the values of Bézier control points over a_1, a_3, a_5 and therefore no function of F_0, F_1, F_2 can be omitted.

Bézier control points of F_0 satisfy:

$$\begin{aligned}
0 &= \lambda_{20}b_6^0 + \lambda_{21}0 + \lambda_{22}b_5^0 \\
0 &= \lambda_{20}0 + \lambda_{21}b_6^0 + \lambda_{22}b_1^0 \\
0 &= \lambda_{10}0 + \lambda_{11}b_1^0 + \lambda_{12}b_2^0 \\
0 &= \lambda_{10}b_2^0 + \lambda_{11}b_3^0 + \lambda_{12}0
\end{aligned}$$

Let b_1^0 is the control point of all non-zero ones of F_0, which is arbitrarily chosen, others are determined by C^1 continuity (see Corollary 7.15):

$$b_3^0 = \frac{\lambda_{10}}{\lambda_{12}}b_1^0, \quad b_5^0 = \frac{\lambda_{20}}{\lambda_{21}}b_1^0.$$

Similar situation appears also for the rest functions F_1, F_2. Let b_5^1 is the arbitrarily chosen control point of F_1. Then

$$b_1^1 = \frac{\lambda_{21}}{\lambda_{20}}b_5^1, \quad b_3^1 = \frac{\lambda_{01}}{\lambda_{02}}b_5^1.$$

And for F_2 let b_3^2 is arbitrarily chosen. Then

$$b_5^2 = \frac{\lambda_{02}}{\lambda_{01}}b_3^2, \quad b_1^2 = \frac{\lambda_{12}}{\lambda_{10}}b_3^2.$$

The condition $b_j = d_0b_j^0 + d_1b_j^1 + d_2b_j^2;\ j = 1, 3, 5$ gives the system of three equations:

$$d_0b_1^0 + d_1\frac{\lambda_{21}}{\lambda_{20}}b_5^1 + d_2\frac{\lambda_{12}}{\lambda_{10}}b_3^2 \quad = \quad b_1$$

$$\begin{aligned} d_0\frac{\lambda_{10}}{\lambda_{12}}b_1^0 + d_1\frac{\lambda_{01}}{\lambda_{02}}b_5^1 + d_2 b_3^2 &= b_3 \\ d_0\frac{\lambda_{20}}{\lambda_{21}}b_1^0 + d_1 b_5^1 + d_2\frac{\lambda_{02}}{\lambda_{01}}b_3^2 &= b_5 \end{aligned}$$

The determinant of the system is

$$D = b_1^0 b_5^1 b_3^2 \begin{vmatrix} 1 & \frac{\lambda_{21}}{\lambda_{20}} & \frac{\lambda_{12}}{\lambda_{10}} \\ \frac{\lambda_{10}}{\lambda_{12}} & \frac{\lambda_{01}}{\lambda_{02}} & 1 \\ \frac{\lambda_{20}}{\lambda_{21}} & 1 & \frac{\lambda_{02}}{\lambda_{01}} \end{vmatrix} = b_1^0 b_5^1 b_3^2 (2 - \lambda - \frac{1}{\lambda}),$$

where $\lambda = \frac{\lambda_{01}}{\lambda_{02}}\frac{\lambda_{12}}{\lambda_{10}}\frac{\lambda_{20}}{\lambda_{21}}$.

$$D = 0 \quad \text{iff} \quad \lambda - 2 + \frac{1}{\lambda} = 0,$$

but it means that $\lambda = 1$ and it is according to (17) impossible. ■

Lemma 7.26 *$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent set of functions.*

Proof: $\operatorname{span}\mathcal{B}_1 \cap \operatorname{span}\mathcal{B}_2 = \emptyset$. Really, all functions in $\mathcal{B}_1$ have zero Bézier control point over the midpoint of each edge in its domain. On the other hand every function in $\mathcal{B}_2$ have at least one of these points non-zero, thus it cannot belong to $\operatorname{span}\mathcal{B}_1$. Similarly all functions in $\mathcal{B}_2$ have zero Bézier point associated with each vertex of the domain unlike the functions in $\mathcal{B}_1$. ■

Theorem 7.27 *$\mathcal{B}$ is a basis of $S_4^1(\mathcal{T})$.*

Proof: From the previous lemma, $\mathcal{B}$ is linearly independent. For the cardinality of $\mathcal{B}$ holds

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_1| + |\mathcal{B}_2| = 3V + V_B + E = \\ &= 3V + V_B + 3V - V_B + 3(N-2) = \\ &= 6V + 3(N-2) = \\ &= \dim(S_4^1(\mathcal{T})) \end{aligned}$$

■

7.6 Minimality

Definition 7.28 *The subset Ω_j of Ω is called **minimal support**, if $\exists F_j \in S_4^1(\mathcal{T})$ such that $\operatorname{supp} F_j = \Omega_j$ and for each non-zero $F \in S_4^1(\mathcal{T})$ such that $\operatorname{supp} F \subset \Omega_j$ holds that $\operatorname{supp} F = \Omega_j$. We will say, that the function F_j **has minimal support**.*

In the end of the work we would like to say few words about the minimality of support of our basis. Again, we will return to the domain with no boundary. Moreover, for all $t \in \mathcal{T}$ we require that $\deg t \geq 5$.

Lemma 7.29 *Let Ω_j is the star of interior vertex $t_j \in \mathcal{T}$. Then Ω_j is the minimal support.*

Proof: Let $E \in S_4^1(\mathcal{T})$; $\operatorname{supp} E \subset \Omega_j, E|_{\Omega_j} \in \tilde{S}_4^1(\mathcal{T}_j)$. Let $\triangle_i = \triangle t_j t_i t_{i+1}$ does not belong to the support of E_j, i.e. $E_j(t) = 0$ for $t \in \triangle_i$. Then all Bézier control points over $\triangle_i$ are zero. Then, because of C^1 continuity along the edge $[t_j t_i]$ also all control points over the triangle $\triangle_{i+1}$ of the line neighbouring to the edge $[t_j t_{i+1}]$ are zero. Also the control points over the boundary of Ω_j and of the neighbouring line are zero. The rest control points over the center of $[t_j t_{i+2}]$ is zero because of C^1 continuity along this edge. So $\triangle_{i+1}$ also does not belong to the support of E and so on. Therefore E is zero function. ■

Lemma 7.30 *Let $t_0, t_1, t_2 \in \mathcal{T}$ are interior with degree at least 5. Let $\triangle t_0 t_1 t_2 \in \mathcal{T}$. Let Ω_{012} consists of triangles $\triangle t_0 t_1 t_2$, $\triangle \tilde{t}_0 t_1 t_2$, $\triangle t_0 \tilde{t}_1 t_2$, $\triangle t_0 t_1 \tilde{t}_2$. Then $\dim(\tilde{S}_4^1(\mathcal{T}_{012})) = 0$.*

Proof: Let us try to find a function F from $S_4^1(\mathcal{T})$ such that supp $F = \Omega_{012}$ and $F|_{\Omega_{012}} \in \tilde{S}_4^1(\mathcal{T}_{012})$.

Because of the condition put on $\tilde{S}_4^1(\mathcal{T}_{012})$, 2 rows of Bézier control points along $\partial\Omega_{012}$ contain only zero points. There are only 6 points left: $b_1, \ldots, b_6$ such that we do not know, whether they can be non-zero.

Let $\tilde{t}_i = \lambda_{i0} t_0 + \lambda_{i1} t_1 + \lambda_{i2} t_2$.

Because of C^1 continuity along the edge $[t_2, t_0]$ the points b_1, b_2, b_3 satisfy:

$$\begin{aligned} 0 &= \lambda_{10} 0 + \lambda_{11} b_1 + \lambda_{12} b_2 \\ 0 &= \lambda_{10} b_2 + \lambda_{11} b_3 + \lambda_{12} 0 \end{aligned}$$

and so on. The matrix of this set of equations is as follows:

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{10} & \lambda_{11} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{00} & \lambda_{01} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{02} & \lambda_{00} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{22} & \lambda_{20} \\ \lambda_{22} & 0 & 0 & 0 & 0 & \lambda_{21} \end{pmatrix},$$

which is equivalent to the following triangular matrix:

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{10} & \lambda_{11} & 0 & 0 & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & 0 & \lambda_{22} & \lambda_{20} \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

where

$$\lambda = \lambda_{21} - \frac{\lambda_{22}\lambda_{12}\lambda_{11}\lambda_{01}\lambda_{00}\lambda_{20}}{\lambda_{11}\lambda_{10}\lambda_{00}\lambda_{02}\lambda_{22}} = \lambda_{21} - \frac{\lambda_{12}\lambda_{01}\lambda_{20}}{\lambda_{10}\lambda_{02}}$$

If $\lambda \neq 0$ then only trivial solution exists and so the rest control points satisfy $b_1 = \ldots = b_6 = 0$.

In our case, $\lambda = 0$ iff

$$\frac{\lambda_{12}}{\lambda_{10}} \frac{\lambda_{01}}{\lambda_{02}} \frac{\lambda_{20}}{\lambda_{21}} = 1.$$

But last equation is never valid according to our assumption (17). ■

Lemma 7.31 *Let t_i, t_j are interior vertices from $\mathcal{T}$, $[t_i t_j] \in \mathcal{T}$ with degree at least 5. Let $\Omega_{ij} \subset \Omega$ is the star of $[t_i t_j]$. Then Ω_{ij} is the minimal support for such functions, which Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: Let the vertices of the star of $[t_i, t_j]$ are denoted as in Figure 16. Let $F \in S_4^1(\mathcal{T})$ be such that supp $F \subset \Omega_{ij}$, supp $F \neq \Omega_{ij}$ and $F|_{\Omega_{ij}} \in \tilde{S}_4^1(\mathcal{T}_{ij})$. First, let $\triangle t_i t_j t_0$ does not belong to the support of F. Then all Bézier control points over $\triangle t_i t_j t_1$ are zero. We can simply convince ourselves that then, because of C^1 continuity, all control points over all rest triangles from $\mathcal{T}_{ij}$ are zero and so F is zero function.

Now, let $\triangle t_i t_0 t_1$ does not belong to supp F. Then thanks to C^1 continuity also $\triangle t_j t_2 t_0$ does not belong to supp F. Only the triangle $\triangle t_j t_i t_5$ with 3 neighbouring triangles stay. But according to Lemma 7.30, it is too small domain to be a support for such function and so F is zero. ■

8 Notes About $S_2^1(\mathcal{T})$

In this section we will focus only on a very special class of $S_2^1(\mathcal{T})$. We are not looking for its basis and are not even calculating its dimension.

Let Ω is the domain and let its triangulation $\mathcal{T}$ looks like a net of regular triangles. Let $f \in S_4^1(\mathcal{T})$. We will discuss the support of f.

Let us try to suppose that the support of f is bounded. Let Ω_1 is such subset of Ω that

$$\begin{aligned}&\partial\Omega \cup \partial\Omega_1 = \emptyset\\&\operatorname{supp} f \subset \Omega_1\end{aligned}$$

Then because of C^1 continuity $f(t) = 0$ and also $Df(t) = 0$ for each $t \in \partial\Omega_1$

There is an example of such Ω_1 on Figure 18.

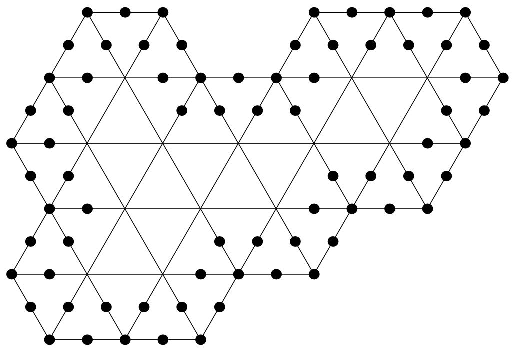

Figure 18: A triangulated subset of the support of a function from $S_2^1(\mathcal{T})$. The points marked by big circle have zero value because of C^1 continuity along the boundary.

We know that control points defined over the boundary of Ω_1 are surely zero because of C^0 continuity along the boundary. Further, the C^1 continuity of f forces all control points of the second row, neighbouring to the boundary one, to be zero. Let v_1 denotes the most upper vertex such that the control point defined over v_1 is not set to zero yet. If there are several such vertices (lying in the same horizontal line), then v_1 is the most left of them. Then if the vertex to the right of v_1 is non-zero, we will denote it v_2, if it is zero, we will skip it and continue with its right neighbour and so on. So in the end we have a horizontal line (not inevitably continuous) of vertices $v_1, \ldots, v_k$, that all control points defined over them are non-zero.

The C^1 continuity around the vertex v_1 forces the control point defined over it to be zero and similarly all neighbouring control points, which are not set to zero yet, will be set to zero now. The same situation repeats for v_2 and so on. After setting the points over $v_1, \ldots, v_k$ and their neighbours to zero, we are in the similar situation as we were in the beginning, only the set, that the support of f is its subset, is made smaller: all triangles over the horizontal line determined by v_1 are put away. We can see that the support of f is the subset of this new, diminished set.

After repeating the featured algorithm several times we find out that the support of f is empty set.

To sum up this section, from the assumption that the support of $f \in S_2^1(\mathcal{T})$ is bounded and situated inside the domain (not at its boundary) we reached the conclusion that f has no support, i.e. it is zero function. It implies, that it is not possible to locally control the C^1 continuous quadratic surface over Ω. The space $S_2^1(\mathcal{T}_1)$ seems to be "too small" for it.

9 Conclusion and Possible Future Work

I presented the question of modeling of C^1 continuous surfaces over the input triangulation. First, the dimension of the space of splines of degree at least 4 was computed and then, using this notion I tried to find a basis of this space but only for the fourth degree splines. I consider this my trial to be quite successful: for enough general input triangulation (I excluded only some special cases) the basis is found. And if additional conditions regarding the degree of vertices are satisfied then even each basis function from the presented basis has minimal support, which in other words means that if the support is made smaller, the function becomes zero.

There are several possibilities how to continue with the research. The problem of "bad" triangulations still remains outstanding. For example, I think that it would be interesting to investigate how will the found basis look like in case the triangulation is close to the special case, that it is not possible to construct such basis for it. Maybe, a type of non-stability will appear.

Further, to find a solution for special cases would be convenient, i.e. to try to find a basis for such triangulations that the algorithm based on this work will fail for them.

So, the question of the basis for the space of C^1 continuous spline functions of fourth degree over triangulation is satisfactory solved. The solution for the functions of higher degree is much simpler and it can be found in the till now published literature (see e.g. already mentioned [Dav98]). The case of C^1 continuous function of second degree is in fact not interesting, because it is impossible to locally control it as was shown in the chapter 8. Only the space of C^1 continuous cubic spline functions over the triangulation still remains to be investigated. It seems, however, to be a tougher task. Meanwhile I do not know even the dimension of this space.

And in the end I am interested in the results which would be brought by the application of polar forms on the found basis. Like in the case of B-spline curves each control point is the value of the belonging polar form where as the arguments several sequential knots of the knot vector appear, also in this case I expect something similar, that the control points of the surface w.r.t. the found basis are values of the polar form of spline surface, where in the place of the arguments the suitable group of vertices of the triangulation appears.

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