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An interpolating 4-point C^2 ternary stationary subdivision scheme

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Abstract

A novel 4-point ternary interpolatory subdivision scheme with a tension parameter is analyzed. It is shown that for a certain range of the tension parameter the resulting curve is C^2. The role of the tension parameter is demonstrated by a few examples. There is a brief discussion of computational costs.

Keywords: Subdivision; Interpolation; Curves; Ternary

1. Introduction

Subdivision schemes have become important in recent years because they provide a uniform and efficient way to describe smooth curves and surfaces. Their beauty lies in the elegant mathematical formulation and simple implementation. Our motivation here is to explore the trade-offs between the degree of continuity of the limit function and the template width for interpolating schemes.

Dubuc (1986), and independently Dyn, Levin and Gregory (1987), describe a 4-point binary interpolating scheme, which they prove to be C^1. Weissman (1990) describes a 6-point binary interpolating scheme that is C^2. Deslauriers and Dubuc (1989) analyse b-ary $2N$-point schemes derived from polynomial interpolation.

Recently Kobbelt introduced a so-called $\sqrt{3}$ scheme (Kobbelt, 2000), which reproduces a ternary scheme after two subdivision steps. The boundary of this class of schemes will reproduce a ternary univariate scheme.

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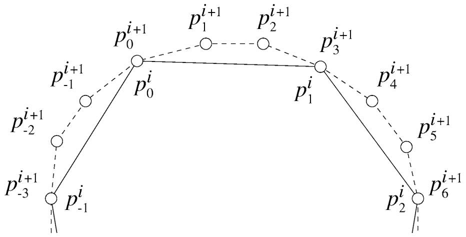

Fig. 1. Ternary scheme. A polygon $P^i = (p_j^i)$ (solid lines) is mapped to a refined polygon $P^{i+1} = (p_j^{i+1})$ (dashed lines). Note that this is an interpolatory scheme: $p_{3j}^{i+1} = p_j^i$.

Here we present an interpolating 4-point ternary univariate stationary subdivision scheme. A polygon $P^i = (p_j^i)$ (see Fig. 1) is mapped to a refined polygon $P^{i+1} = (p_j^{i+1})$ by applying the following three subdivision rules:

$$\begin{aligned}
p_{3j}^{i+1} &= p_j^i, \\
p_{3j+1}^{i+1} &= a_0 p_{j-1}^i + a_1 p_j^i + a_2 p_{j+1}^i + a_3 p_{j+2}^i, \\
p_{3j+2}^{i+1} &= a_3 p_{j-1}^i + a_2 p_j^i + a_1 p_{j+1}^i + a_0 p_{j+2}^i,
\end{aligned} \tag{1}$$

where the weights $\{a_i\}$ are given by

$$\begin{aligned}
a_0 &= -\frac{1}{18} - \frac{1}{6}\mu, \\
a_1 &= \frac{13}{18} + \frac{1}{2}\mu, \\
a_2 &= \frac{7}{18} - \frac{1}{2}\mu, \\
a_3 &= -\frac{1}{18} + \frac{1}{6}\mu.
\end{aligned} \tag{2}$$

We can see immediately from this that

$$a_0 + a_1 + a_2 + a_3 = 1. \tag{3}$$

These weights were the solutions of a constraint problem derived from the constant, linear, and quadratic precision conditions, which are necessary for C^2-continuity.

In the following section we look at the support for this scheme. In Sections 3 and 4, we analyze the limit function and prove that it is C^2 for $\frac{1}{15} < \mu < \frac{1}{9}$. Then we find the precision set for this scheme and illustrate the effects of μ using some specific examples. A brief discussion of computational costs follow, and finally we suggest some further work.

2. Support

It is necessary to calculate the support for this scheme before we can do the analysis that follows. First consider all the vertices lying on an axis. This means that all the new

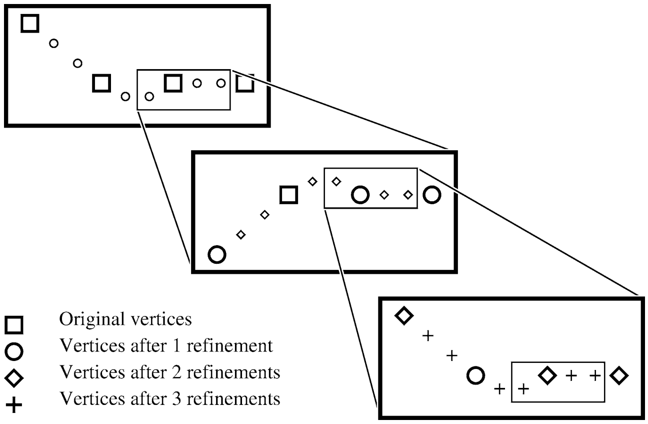

Fig. 2. Illustration of similarity of support at different subdivision levels. As we carry out the subdivision we see that each boxed region is similar to the whole region.

vertices will also lie on this axis. Now we consider the consequences of moving one of the vertices above the axis. At the first subdivision step, we see that the vertex at $5/3$ is the furthest non-zero new vertex. At the next step of the scheme this will propagate along by $5/3 \times 1/3$, by similarity (see Fig. 2). Hence after n subdivisions the furthest non-zero vertex will be at

$$5\sum_{i=1}^{n} \frac{1}{3^i}$$

and hence the total support is

$$2 \times 5\sum_{i=1}^{\infty} \frac{1}{3^i} = 5.$$

This support compares favourably with the the 4-point binary scheme having a support of 6 and the 6-point binary scheme having a support of 10.

Also, as the scheme has negative outer coefficients, it introduces a new zero-crossing, that is a new point lying on the axis with its immediate neighbours above and below the axis, at each step (again, see Fig. 2). As the scheme is also interpolatory, the basis function will have an infinite number of zero-crossings, and hence cannot be described by a curve with a finite number of polynomial pieces.

3. Convergence analysis—necessary conditions

Matrix formalism allows us to derive necessary conditions for a scheme to be C^k based on the eigenvalues of the subdivision matrices.

Suppose the eigenvalues are $\{\lambda_i\}$, where $\lambda_0 = 1$ and $|\lambda_i| \geqslant |\lambda_{i+1}|\ \forall i \in \mathbb{N}$. Then we have the following necessary conditions for the corresponding properties:

$$\begin{aligned}
|\lambda_1| &= |\lambda_2| &&\Leftarrow \text{kink, i.e., not } C^1\\
\lambda_1^2 &< \lambda_2 &&\Leftarrow \text{unbounded curvature}\\
\lambda_1^2 &= |\lambda_2| = |\lambda_3| &&\Leftarrow \text{mildly diverging curvature}\\
\lambda_1^2 &= |\lambda_2| > |\lambda_3| &&\Leftarrow \text{curvature bounded}\\
\lambda_1^2 &> |\lambda_2| &&\Leftarrow \text{curvature} \to 0
\end{aligned} \tag{4}$$

This analysis was first demonstrated in (Doo and Sabin, 1978), in which the terminology is explained more fully. For the purposes of this paper we are interested in showing that $\lambda_1^2 = |\lambda_2| > |\lambda_3|$, i.e., that the curvature of the limit function is bounded, which is a necessary condition if the limit function is to have C^2-continuity.

We shall perform the analysis for the mark points of this scheme. The mark points are the points which are topologically invariant under the subdivision step. For this scheme, the mark points are the mid-point between two vertices and the vertices themselves.

3.1. Mid-point

In this analysis we need consider only three vertices on either side of the mid-point, because the support tells us that the vertices lying further than this have no effect at the point we wish to analyze. So consider the original vertices $\{A, B, C, D, E, F\}$ and the new vertices $\{a, b, c, d, e, f\}$ in Fig. 3. We have from (1)

$$\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f \end{pmatrix} = \begin{pmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & a_0 & a_1 & a_2 & a_3 & 0\\ 0 & a_3 & a_2 & a_1 & a_0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} A\\ B\\ C\\ D\\ E\\ F \end{pmatrix}. \tag{5}$$

The eigenvalues for this are $1, \frac{1}{3}, \frac{1}{9}, \mu, -\frac{1}{18} + \frac{1}{6}\mu$ (twice).

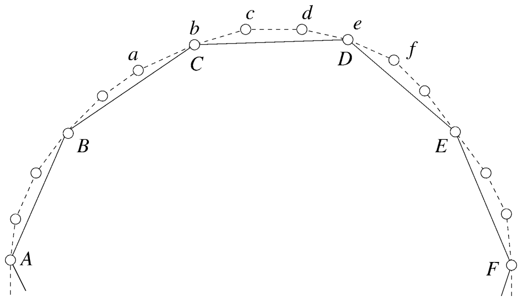

Fig. 3. Configuration around mid-point.

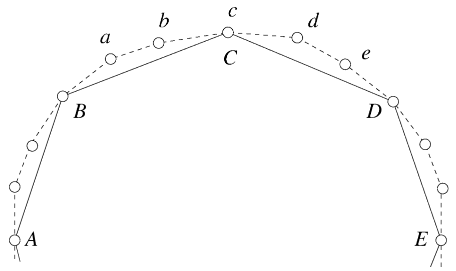

Fig. 4. Configuration around vertex.

3.2. Vertex

We now wish to calculate the eigenvalues of the vertex subdivision matrix. This time we need consider only the two vertices on each side of the vertex, because the support tells us that the vertices lying further than this have no effect at the point we wish to analyze. So consider the vertices $\{A, B, C, D, E\}$ and the new vertices $\{a, b, c, d, e\}$ in Fig. 4. We have from (1)

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix}. \tag{6}$$

The eigenvalues are 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{18} - \frac{1}{2}\mu$, $\frac{1}{6} - \frac{5}{6}\mu$.

3.3. Bounds on μ

The two subdivision matrices and (4) allow us to find bounds on μ which are necessary for the limit function to be C^2. The mid-point subdivision matrix satisfies the necessary conditions for C^2 iff

$$|\mu| < \frac{1}{9}. \tag{7}$$

Moreover the necessary conditions for C^3 cannot be satisfied for this matrix. For the range of μ in (7), $|\frac{1}{6} - \frac{5}{6}\mu| > |\frac{1}{18} - \frac{1}{2}\mu|$. Hence we see that the necessary conditions for C^2 are satisfied by the vertex subdivision matrix iff

$$\left|\frac{1}{6} - \frac{5}{6}\mu\right| < \frac{1}{9}. \tag{8}$$

(7) and (8) are both satisfied iff

$$\frac{1}{15} < \mu < \frac{1}{9}. \tag{9}$$

This is illustrated by Fig. 5. From this we can also see that when $\mu = \frac{1}{11}$ we get the best trade-off in magnitude between the fourth largest eigenvalues of vertex and mid-point subdivision matrices.

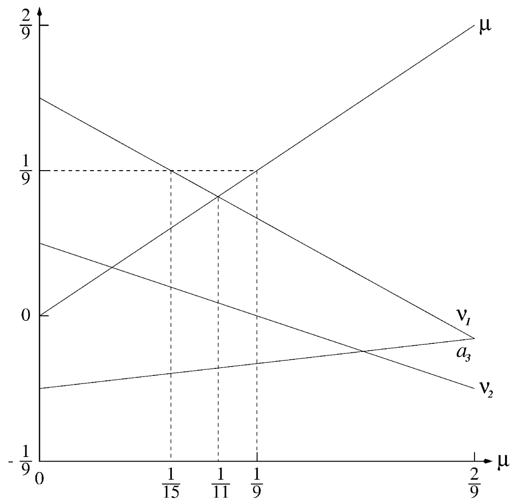

Fig. 5. Plot of μ-dependant eigenvalues against μ. μ and a_3 are the eigenvalues of the mid-point subdivision matrix, and ν_1 and ν_2 are the eigenvalues of the vertex subdivision matrix. $a_3 = -\frac{1}{18} + \frac{1}{6}\mu$, $\nu_1 = \frac{1}{6} - \frac{5}{6}\mu$, $\nu_2 = \frac{1}{18} - \frac{1}{2}\mu$.

Hence the necessary conditions for the limit function of the scheme to be C^2 are satisfied for both mark points for μ in the range above. For C^2 continuity there are other necessary conditions on the eigenvectors (Warren, t.a.). One can verify from the appendices that these conditions are also violated at the two extremes of the value μ.

4. Convergence analysis—sufficient conditions

The generating function formalism lends itself well to deriving sufficient conditions for subdivision schemes to be C^k. For this scheme the subdivision step (2) can be compactly written in a single equation

$$p_j^{i+1} = \sum_{k \in \mathbb{Z}} \alpha_{3k-j} p_k^i, \tag{10}$$

where

$$\alpha = (\alpha_j) = \left[\ldots, 0, 0, a_3, a_0, 0, a_2, a_1, 1, a_1, a_2, 0, a_0, a_3, 0, 0, \ldots\right]. \tag{11}$$

From this we can see immediately that

$$\sum_{j \in \mathbb{Z}} \alpha_{3j} = 1, \quad \sum_{j \in \mathbb{Z}} \alpha_{3j+1} = 1, \quad \sum_{j \in \mathbb{Z}} \alpha_{3j+2} = 1. \tag{12}$$

After some computation (Dyn, 1992) we see that the subdivision step can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z) P^i\left(z^3\right), \tag{13}$$

where

$$P^i(z) = \sum_j p^i_j z^j, \quad \alpha(z) = \sum_j \alpha_j z^j. \tag{14}$$

4.1. Sufficient conditions for C^k

Now we will go on to derive sufficient conditions for a ternary scheme to be C^k and use this to show that the scheme we have proposed is in fact C^2. The proof is very similar to that given by Dyn (1992) for a binary scheme.

Proposition 4.1. *Let S be a subdivision scheme defined by a mask satisfying* (12). *Then there exists a subdivision scheme S_1 with the property*

$$dP^{i+1} = S_1 dP^i, \tag{15}$$

where $P^i = S^i P^0$, and $(dP^i)_j = 3^i(p^i_{j+1} - p^i_j)$.

Proof. Let Λ denote the set of all Laurent polynomials and define the characteristic Λ-polynomial of S by $\alpha(z)$. Then by (12)

$$\alpha(1) = 3, \quad \alpha\big(e^{2i\pi/3}\big) = 0, \quad \alpha\big(e^{4i\pi/3}\big) = 0. \tag{16}$$

Therefore

$$\alpha'(z) = \frac{3z^2(1-z)\alpha(z)}{1-z^3} \in \Lambda. \tag{17}$$

We now show that the mask determined by $\alpha'(z)$ defines a subdivision scheme S_1 with the required properties. Defining $H^i(z)$ to be the symbol of dP^i, we get

$$\begin{aligned} H^i(z) &= \sum_{j\in\mathbb{Z}} \big(dP^i\big)_j z^j \quad \text{(definition)} \\ &= 3^i \sum_{j\in\mathbb{Z}} \big(p^i_{j+1} - p^i_j\big) z^j = 3^i\big(z^{-1}P^i(z) - P^i(z)\big). \end{aligned} \tag{18}$$

Hence,

$$H^i(z) = 3^i P^i(z)\frac{1-z}{z}, \tag{19}$$

and by application of (13) we get

$$H^{i+1}(z) = 3^{i+1}P^{i+1}(z)\frac{1-z}{z} = 3^{i+1}\alpha(z)P^i\big(z^3\big)\frac{1-z}{z}. \tag{20}$$

Thus, by (19)

$$H^{i+1}(z) = 3\alpha(z)H^i\big(z^3\big)\frac{z^3}{1-z^3}\frac{1-z}{z} = \alpha'(z)H^i\big(z^3\big), \tag{21}$$

a relation similar in form to (13). Recalling the definition of $H^i(z)$, we conclude the existence of a subdivision scheme S_1 satisfying (17) with a mask determined by the characteristic Λ-polynomial

$$\alpha'(z) = \frac{3z^2(1-z)\alpha(z)}{(1-z^3)}. \qquad \square \tag{22}$$

We can now determine the convergence of S by analyzing the subdivision scheme $\frac{1}{3}S_1$.

Theorem 4.2. *S is a uniformly convergent subdivision scheme, if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data P^0.*

Proof. See proof of Theorem 3.2 in (Dyn, 1992). $\square$

Theorem 4.2 indicates that for any given subdivision scheme, S, with a mask α satisfying (12), we can prove the uniform convergence of S by first deriving the mask of $\frac{1}{3}S_1$ and then computing $\|(\frac{1}{3}S_1)^i\|_\infty$ for $i = 1, 2, 3, \ldots, L$, where L is the first integer for which $\|(\frac{1}{3}S_1)^L\|_\infty < 1$. If such an L exists, S converges uniformly.

Theorem 4.3. *If S is a uniformly convergent subdivision scheme, then it determines a unique compactly supported continuous function $S^\infty P^0$.*

Proof. See proof of Theorem 2.5 in (Dyn, 1992). $\square$

Here $S^n P^0$ is the scheme applied n times to the initial polygon P^0, hence $S^\infty P^0$ is the limit function. Once the uniform convergence of S is established, we are then interested in determining the smoothness of the limit function $S^\infty P^0$.

Theorem 4.4. *Let S be a subdivision scheme with a characteristic Λ-polynomial*

$$\alpha(z) = \left(\frac{1-z^3}{3z^2(1-z)}\right)^k q(z), \quad q \in \Lambda. \tag{23}$$

If the subdivision scheme S_k, corresponding to the Λ-polynomial $q(z)$, converges uniformly then $S^\infty P^0 \in C^k$ for any initial control polygon P^0.

Proof. See proof of Theorem 3.4 in (Dyn, 1992). $\square$

Corollary 4.5. *If S is a subdivision scheme of the form above and $\frac{1}{3}S_{k+1}$ converges uniformly to the zero function for all initial data P^0 then $S^\infty P^0 \in C^k$ for any initial control polygon P^0.*

Proof. Apply Theorem 4.2 to Theorem 4.4. $\square$

Corollary 4.5 indicates that for any given ternary subdivision scheme, S, we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{3}S_{k+1}$ and then computing $\|(\frac{1}{3}S_{k+1})^i\|_\infty$ for

$i = 1, 2, 3, \ldots, L$, where L is the first integer for which $\|(\frac{1}{3}S_{k+1})^L\|_\infty < 1$. If such an L exists, $S^\infty P^0 \in C^k$.

4.2. *Proof of* C^2

For our scheme we have

$$\alpha = \frac{1}{18}\big[\ldots, 0, 0, 3\mu - 1, -3\mu - 1, 0, -9\mu + 7, 9\mu + 13, 18, \\ 9\mu + 13, -9\mu + 7, 0, -3\mu - 1, 3\mu - 1, 0, 0, \ldots\big], \tag{24}$$

$$\alpha^{(1)} = \frac{1}{6}\big[\ldots, 0, 0, 3\mu - 1, -6\mu, 3\mu + 1, -6\mu + 6, 12\mu + 6, \\ -6\mu + 6, 3\mu + 1, -6\mu, 3\mu - 1, 0, 0, \ldots\big], \tag{25}$$

$$\alpha^{(2)} = \frac{1}{2}\big[\ldots, 0, 0, 3\mu - 1, -9\mu + 1, 9\mu + 1, -6\mu + 4, \\ 9\mu + 1, -9\mu + 1, 3\mu - 1, 0, 0, \ldots\big], \tag{26}$$

$$\alpha^{(3)} = \frac{3}{2}\big[\ldots, 0, 0, 3\mu - 1, -12\mu + 2, 18\mu, -12\mu + 2, 3\mu - 1, 0, 0, \ldots\big]. \tag{27}$$

The calculation of $\alpha^{(4)}$ does not give us a Laurent polynomial. It is easy to verify that $\alpha, \alpha^{(1)}, \alpha^{(2)}$, all satisfy (12). Using

$$\left\|\frac{1}{3}S_{k+1}\right\|_\infty = \frac{1}{3}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha_{3j}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+1}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+2}^{(k+1)}\right|\right) \tag{28}$$

for $\frac{1}{15} < \mu < \frac{1}{9}$, we have

$$\left\|\frac{1}{3}S_1\right\|_\infty = \frac{4\mu + 1}{3} < 1, \tag{29}$$

$$\left\|\frac{1}{3}S_2\right\|_\infty = -2\mu + 1 < 1, \tag{30}$$

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left(9\mu, \frac{-15\mu + 3}{2}\right) < 1. \tag{31}$$

Hence all the sufficient conditions are satisfied for this scheme to be C^2. □

We note that the same range of μ occurs both in the sufficient and necessary conditions and so cannot be improved.

Now that we have derived the continuity and smoothness properties of this scheme we can look at other properties. First we look at the precision set.

5. Precision set

If we have three points, p_0, p_1, p_2, we can fit a quadratic through them, as follows:

$$P(t) = \frac{t}{2}(t - 1)p_0 + \left(1 - t^2\right)p_1 + \frac{t}{2}(t + 1)p_2 \tag{32}$$

such that

$$P(-1) = p_0, \quad P(0) = p_1, \quad P(1) = p_2. \tag{33}$$

Now, if we define

$$p_3 = P(2) = p_0 - 3p_1 + 3p_2 \tag{34}$$

we have four vertices with which to carry out a subdivision step using this scheme. The new vertices are

$$\begin{aligned} p_1^1 &= \left(-\frac{1}{18} - \frac{1}{6}\mu\right)p_0 + \left(\frac{13}{18} + \frac{1}{2}\mu\right)p_1 + \left(\frac{7}{18} - \frac{1}{2}\mu\right)p_2 + \left(-\frac{1}{18} + \frac{1}{6}\mu\right)p_3 \\ &= -\frac{1}{9}p_0 + \frac{8}{9}p_1 + \frac{2}{9}p_2 \\ &= P\left(\frac{1}{3}\right), \end{aligned} \tag{35}$$

$$\begin{aligned} p_2^1 &= \left(-\frac{1}{18} + \frac{1}{6}\mu\right)p_0 + \left(\frac{7}{18} - \frac{1}{2}\mu\right)p_1 + \left(\frac{13}{18} + \frac{1}{2}\mu\right)p_2 + \left(-\frac{1}{18} - \frac{1}{6}\mu\right)p_3 \\ &= -\frac{1}{9}p_0 + \frac{5}{9}p_1 + \frac{5}{9}p_2 \\ &= P\left(\frac{2}{3}\right). \end{aligned} \tag{36}$$

Hence the new vertices lie on the original quadratic.

This means that if we define a set of vertices $\{p_j\}$, where $p_j = P(j),\ j \in \mathbb{Z}$, all the new vertices calculated by this scheme will also lie on this quadratic. Hence the limit function will be this quadratic.

We cannot do the same for a cubic. Hence the precision set for this scheme is the quadratics.

We find that if we put a cubic through the general points A, B, C, D, such that $Q(-1) = A,\ Q(0) = B,\ Q(1) = C,\ Q(2) = D$

$$\begin{aligned} Q(t) &= \frac{t(1-t)(t-2)}{6}A + \frac{(t+1)(t-1)(t-2)}{2}B \\ &\quad + \frac{t(t+1)(2-t)}{2}C + \frac{t(t+1)(t-1)}{3}D \end{aligned} \tag{37}$$

and calculate $Q(1/3), Q(2/3)$ we recover the coefficients for the ternary 4-pt Dubuc–Deslaurier scheme (Deslauriers and Dubuc, 1989). If we calculate $Q(1/2)$ we recover the coefficients for the 4-pt binary scheme. Hence both these schemes have cubic precision. However, eigenanalysis shows that both these schemes can have unbounded curvature [1] and hence are only C^1.

[1] At the vertex, Dubuc's ternary scheme has eigenvalues $1, 1/3, 11/81, 1/9, 1/27$, where the third eigenvalue is greater than $1/3^2$, and the 4-pt binary scheme has eigenvalues $1, 1/2, 1/4, 1/4, 1/8$, where the repeated eigenvalue causes mildly divergent curvature.

6. Examples

In order to investigate how the parameter μ affects the limit function, we need to diagonalize the subdivision matrices and look at some specific examples. Diagonalization of the subdivision matrices allows us to raise the matrices to the nth power and we can deduce properties of the limit function by taking the limit as $n \to \infty$. The diagonalizations can be seen in Appendices A and B. The first two examples show, in a sense, the worst-case behaviour of the scheme.

6.1. Basis function

Let us first look at the basis function. This is the limit function of the scheme when applied to the vertices $\{P_i\}$ where $P_i = (i, 0)$, $i \in \mathbb{Z} \setminus \{0\}$, and $P_0 = (0, 1)$. An approximation to this can be seen in Fig. 6 for $\mu = \frac{1}{11}$. The support tells us that all the points beyond two and a half units from the origin will be on the axis.

We can define an approximation to the discrete curvature at a vertex, for a given subdivision step, by calculating the circumcircle of the triangle formed by the vertex and its immediate neighbours. This has been used to produce the curvature plot in Fig. 7.

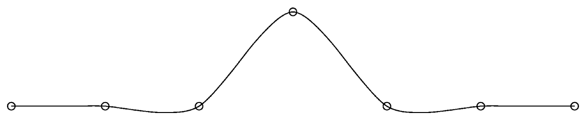

Fig. 6. Result of the scheme for the basis function after 4 subdivision steps with $\mu = \frac{1}{11}$.

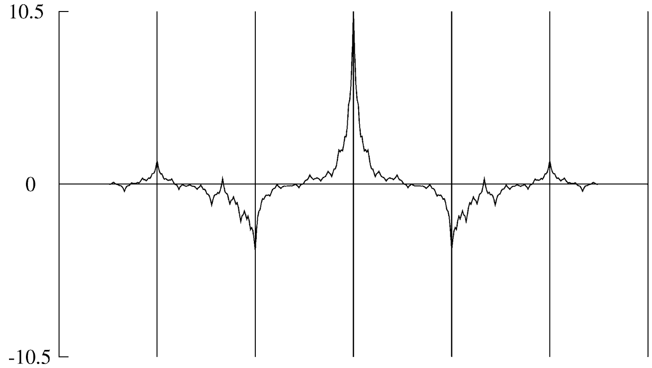

Fig. 7. Curvature plot of Fig. 6. The curvature peaks at the non-zero vertex, but is bounded. Vertical lines are at integer locations, the central line is at $i = 0$.

We observe that the curvature peaks at the non-zero vertex. We can calculate the curvature of the limit function at this point from (A.3). After n subdivision steps the configuration around this vertex will be

$$\mathbf{M}_v^n \begin{pmatrix} -2 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3^n} & 1 - \frac{8(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n + \frac{9(1+\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n \\ -\frac{1}{3^n} & 1 - \frac{2(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n + \frac{3(1-3\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n \\ 0 & 1 \\ \frac{1}{3^n} & 1 - \frac{2(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n + \frac{3(1-3\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n \\ \frac{2}{3^n} & 1 - \frac{8(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n + \frac{9(1+\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n \end{pmatrix}. \tag{38}$$

Calculating the curvature as the inverse of the radius of the circumcircle through the middle 3 points, we get

$$\frac{1}{r} = \frac{2\left(\frac{2(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n - \frac{3(1-3\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n\right)}{\left(\frac{1}{3^n}\right)^2 + \left(\frac{2(1+3\mu)}{-1+15\mu}\left(\frac{1}{9}\right)^n - \frac{3(1-3\mu)}{-1+15\mu}\left(\frac{1}{6} - \frac{5}{6}\mu\right)^n\right)^2} \tag{39}$$

and the curvature of the limit function at the peak is given by

$$\lim_{n\to\infty} \frac{1}{r} = \begin{cases} 4\frac{1+3\mu}{-1+15\mu}, & \mu > \frac{1}{15}, \\ \infty & \text{otherwise.} \end{cases} \tag{40}$$

Fig. 8 plots the peak curvature against μ and from this we can see that the curvature becomes very large as we near $\mu = \frac{1}{15}$. For $\mu = \frac{1}{11}$ the peak curvature is 14. This is the same order of magnitude as that for the basis functions of the overhauser cubic and the cubic spline.

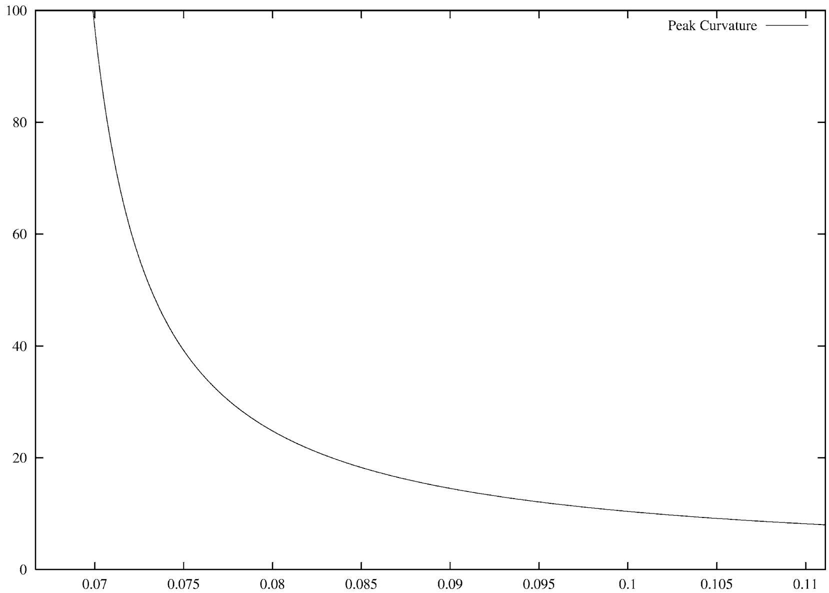

Fig. 8. Peak curvature against μ for the basis function.

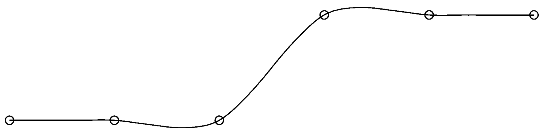

Fig. 9. Result of the scheme for the step function after 4 subdivision steps with $\mu = \frac{1}{11}$.

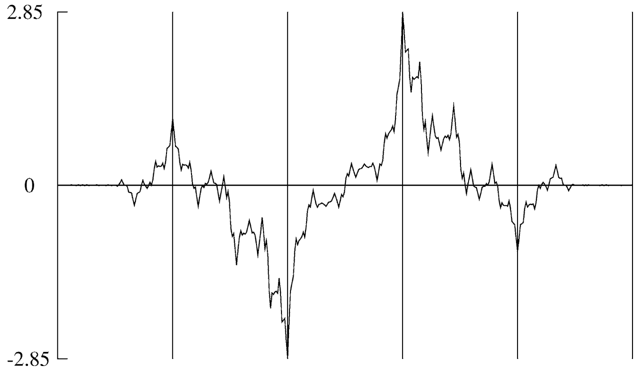

Fig. 10. Curvature plot of Fig. 9.

6.2. Step function

We will now use the step function to illustrate the effect of the upper bound of μ. The step function is the limit function of the scheme when applied to the vertices $\{P_i\}$ where

$$P_i = \begin{cases} (2i+1,-1), & i < 0, \\ (2i+1,1), & i \geqslant 0, \end{cases} \quad i \in \mathbb{Z}. \tag{41}$$

An approximation to this can be seen in Fig. 9 for $\mu = \frac{1}{11}$, and the curvature[2] plot of this can be seen in Fig. 10. We are interested in what happens at the mid-point. It is clear, by symmetry, that the curvature here should be zero. Using (B.3), after n subdivision steps the configuration around the mid-point will be

$$\mathbf{M}_m^n \begin{pmatrix} -5 & -1 \\ -3 & -1 \\ -1 & -1 \\ 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3^n} & -c \\ -\frac{3}{3^n} & -\frac{3^{1-n}(\mu-1)+2\mu^n}{3\mu-1} \\ -\frac{1}{3^n} & -\frac{3^{-n}(\mu-1)+2\mu^{1+n}}{3\mu-1} \\ \frac{1}{3^n} & \frac{3^{-n}(\mu-1)+2\mu^{1+n}}{3\mu-1} \\ \frac{3}{3^n} & \frac{3^{1-n}(\mu-1)+2\mu^n}{3\mu-1} \\ \frac{5}{3^n} & c \end{pmatrix}, \tag{42}$$

where c is an expression that is superfluous to the following analysis.

[2] The curvature is calculated as in the previous section.

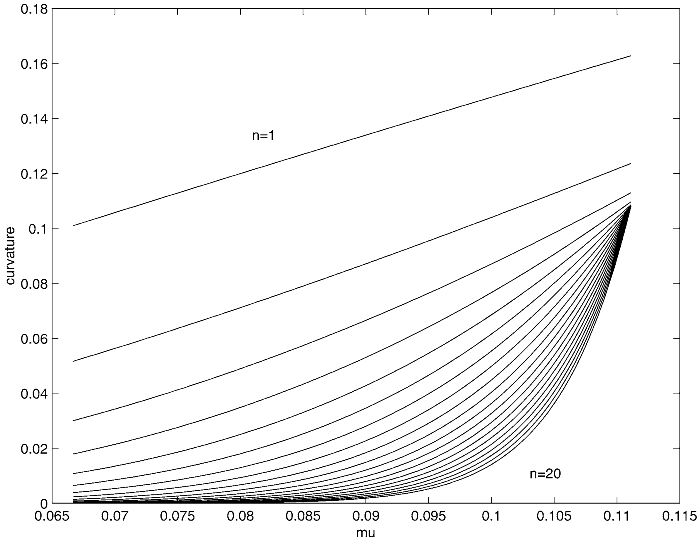

Fig. 11. Curvature at the vertex nearest the mid-point after n subdivision steps as a function of μ.

We can now calculate the curvature at the vertex nearest the mid-point as the inverse of the radius of the circumcircle through the points

$$\begin{aligned}&\left(-\tfrac{1}{3^n}, -\tfrac{3^{-n}(\mu-1)+2\mu^{1+n}}{3\mu-1}\right),\\ &\left(\tfrac{1}{3^n}, \tfrac{3^{-n}(\mu-1)+2\mu^{1+n}}{3\mu-1}\right),\\ &\left(\tfrac{3}{3^n}, \tfrac{3^{1-n}(\mu-1)+2\mu^{n}}{3\mu-1}\right).\end{aligned} \tag{43}$$

This can be seen in Fig. 11. Taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{r} = \begin{cases} 0, & \mu < \frac{1}{9}, \\ \frac{27}{250}, & \mu = \frac{1}{9}. \end{cases} \tag{44}$$

This implies that the curvature of the step function becomes discontinuous at $\mu = \frac{1}{9}$. We have now illustrated how both bounds on μ affect the limit function.

6.3. *Other examples*

For most examples it is very difficult for the naked eye to distinguish how the limit function changes as μ is changed within the allowed range. In Fig. 12 we have used a simple example to illustrate how μ affects the limit function.

7. Computational costs

To produce a curve with C^2 continuity using binary subdivision requires a 6-point scheme (Weissman, 1990).

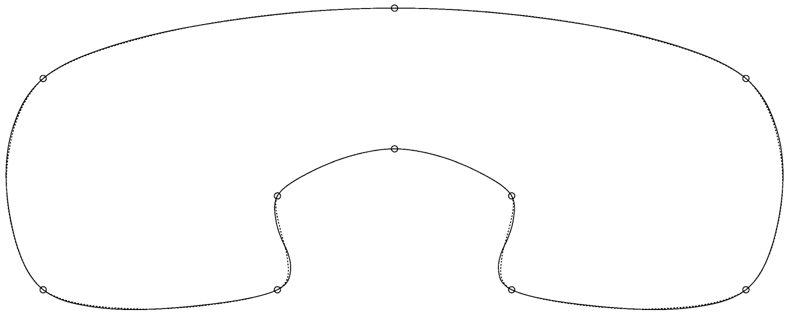

Fig. 12. The result of the scheme after 4 subdivision steps. The solid line is produced by setting $\mu = \frac{1}{9}$. The dashed line is produced by setting $\mu = \frac{1}{15}$.

Each new point generated by the 6-point binary scheme requires 6 multiplies and 5 adds: a total of 11 floating point operations. Each new point generated by a 4-point ternary scheme requires 4 multiplies and 3 adds: a total of 7 floating point operations. However, a ternary subdivision step introduces twice as many new vertices as a binary subdivision step.

After some calculation we have that the number of floating point operations required to produce a discrete approximation with k times as many vertices as an original approximation with n vertices is

$$11\left(2^{\lceil \log_2 k \rceil} - 1\right)n \tag{45}$$

for a 6-point binary scheme and

$$7\left(3^{\lceil \log_3 k \rceil} - 1\right)n \tag{46}$$

for a 4-point ternary scheme.

The two schemes are thus roughly equal in computational cost, with the ternary scheme having an advantage on average, because

$$\frac{7(3^{\log_3 k} - 1)n}{11(2^{\log_2 k} - 1)n} = \frac{7}{11}. \tag{47}$$

We can see from the graph of $\frac{\text{cost}}{n}$ against k for the ternary and binary schemes (Fig. 13) that the ternary scheme has a lower cost than that of the binary scheme for a greater range of k.

8. Further work

We have shown that in univariate interpolating subdivision we can achieve greater smoothness with the same number of control points by using a ternary rather than a binary subdivision scheme. Also, for the same smoothness, the ternary scheme presented in this

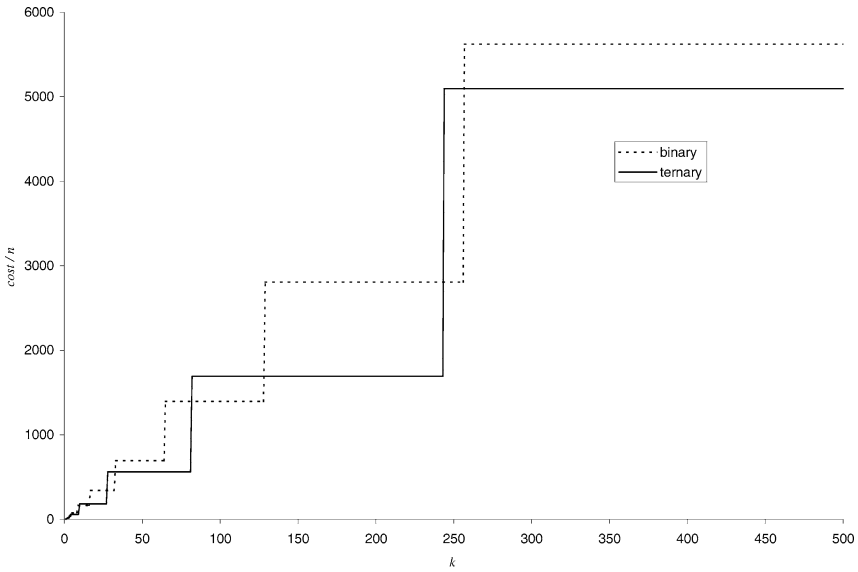

Fig. 13. Graph of $\frac{\text{cost}}{n}$ against k for the ternary and binary schemes. n is the number of original vertices and kn is the number of new vertices.

paper has a much smaller support and slightly lower computational cost than the equivalent binary scheme. [3]

We have presented a 4-point C^2 scheme in this paper and (Hassan and Dodgson, 2001) shows that we can achieve C^1 with a 3-point scheme. It is yet to be investigated whether we can keep increasing the number of new points introduced in each subdivision step to achieve even greater smoothness, i.e., whether a quinary 4-point scheme can yield a C^3 curve, and so on.

Appendix A. Vertex subdivision matrix diagonalization

The subdivision matrix for the vertex (6) can be written

$$\begin{aligned}\mathbf{M}_v &= \begin{pmatrix} -\frac{1}{18}-\frac{1}{6}\mu & \frac{13}{18}+\frac{1}{2}\mu & \frac{7}{18}-\frac{1}{2}\mu & -\frac{1}{18}+\frac{1}{6}\mu & 0 \\ -\frac{1}{18}+\frac{1}{6}\mu & \frac{7}{18}-\frac{1}{2}\mu & \frac{13}{18}+\frac{1}{2}\mu & -\frac{1}{18}-\frac{1}{6}\mu & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{18}-\frac{1}{6}\mu & \frac{13}{18}+\frac{1}{2}\mu & \frac{7}{18}-\frac{1}{2}\mu & -\frac{1}{18}+\frac{1}{6}\mu \\ 0 & -\frac{1}{18}+\frac{1}{6}\mu & \frac{7}{18}-\frac{1}{2}\mu & \frac{13}{18}+\frac{1}{2}\mu & -\frac{1}{18}-\frac{1}{6}\mu \end{pmatrix} \\ &= \mathbf{V}_v\mathbf{D}_v\mathbf{U}_v, \end{aligned} \tag{A.1}$$

where

[3] This is the 6-point scheme presented in (Weissman, 1990).

$$\mathbf{V}_v = \begin{pmatrix} 1 & 2 & 4 & \frac{3}{2}\frac{1+\mu}{-1+15\mu} & \frac{1}{2}\frac{7+3\mu}{5+9\mu} \\ 1 & 1 & 1 & -\frac{1}{2}\frac{-1+3\mu}{-1+15\mu} & -\frac{1}{2}\frac{-1+3\mu}{5+9\mu} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -\frac{1}{2}\frac{-1+3\mu}{-1+15\mu} & \frac{1}{2}\frac{-1+3\mu}{5+9\mu} \\ 1 & -2 & 4 & \frac{3}{2}\frac{1+\mu}{-1+15\mu} & -\frac{1}{2}\frac{7+3\mu}{5+9\mu} \end{pmatrix},$$

$$\mathbf{D}_v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6}-\frac{5}{6}\mu & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{18}-\frac{1}{2}\mu \end{pmatrix},$$

$$\mathbf{U}_v = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2}\frac{-1+3\mu}{5+9\mu} & \frac{1}{2}\frac{7+3\mu}{5+9\mu} & 0 & -\frac{1}{2}\frac{7+3\mu}{5+9\mu} & -\frac{1}{2}\frac{-1+3\mu}{5+9\mu} \\ \frac{1}{2}\frac{-1+3\mu}{-1+15\mu} & \frac{3}{2}\frac{1+\mu}{-1+15\mu} & -2\frac{1+3\mu}{-1+15\mu} & \frac{3}{2}\frac{1+\mu}{-1+15\mu} & \frac{1}{2}\frac{-1+3\mu}{-1+15\mu} \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \end{pmatrix}, \tag{A.2}$$

and $\mathbf{U}_v\mathbf{V}_v = \mathbf{I}$. Hence

$$\mathbf{M}_v^n = \mathbf{V}_v\mathbf{D}_v^n\mathbf{U}_v. \tag{A.3}$$

Appendix B. Mid-point subdivision matrix diagonalization

The subdivision matrix for the mid-point (5) can be written

$$\begin{aligned}\mathbf{M}_m &= \begin{pmatrix} -\frac{1}{18}+\frac{1}{6}\mu & \frac{7}{18}-\frac{1}{2}\mu & \frac{13}{18}+\frac{1}{2}\mu & -\frac{1}{18}-\frac{1}{6}\mu & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{18}-\frac{1}{6}\mu & \frac{13}{18}+\frac{1}{2}\mu & \frac{7}{18}-\frac{1}{2}\mu & -\frac{1}{18}+\frac{1}{6}\mu & 0 \\ 0 & -\frac{1}{18}+\frac{1}{6}\mu & \frac{7}{18}-\frac{1}{2}\mu & \frac{13}{18}+\frac{1}{2}\mu & -\frac{1}{18}-\frac{1}{6}\mu & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{18}-\frac{1}{6}\mu & \frac{13}{18}+\frac{1}{2}\mu & \frac{7}{18}-\frac{1}{2}\mu & -\frac{1}{18}+\frac{1}{6}\mu \end{pmatrix} \\ &= \mathbf{V}_m\mathbf{D}_m\mathbf{U}_m,\end{aligned} \tag{B.1}$$

where

$$\mathbf{V}_m = \begin{pmatrix} 1 & 5 & 25 & 12\mu^2+5\mu+7 & \frac{1}{2} & \frac{1}{2} \\ 1 & 3 & 9 & 15\mu+1 & 0 & 0 \\ 1 & 1 & 1 & \mu(15\mu+1) & 0 & 0 \\ 1 & -1 & 1 & -\mu(15\mu+1) & 0 & 0 \\ 1 & -3 & 9 & -15\mu-1 & 0 & 0 \\ 1 & -5 & 25 & -12\mu^2-5\mu-7 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\mathbf{D}_m = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{18}+\frac{1}{6}\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{18}+\frac{1}{6}\mu \end{pmatrix},$$

$$\mathbf{U}_m = \begin{pmatrix} 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & \frac{\mu}{2(3\mu-1)} & -\frac{1}{2(3\mu-1)} & \frac{1}{2(3\mu-1)} & -\frac{\mu}{2(3\mu-1)} & 0 \\ 0 & \frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 \\ 0 & -\frac{1}{2(45\mu^2-12\mu-1)} & \frac{3}{2(45\mu^2-12\mu-1)} & -\frac{3}{2(45\mu^2-12\mu-1)} & \frac{1}{2(45\mu^2-12\mu-1)} & 0 \\ 1 & -3 & 2 & 2 & -3 & 1 \\ \frac{15\mu+1}{15\mu+1} & \frac{-21\mu-7}{15\mu+1} & \frac{-12\mu+16}{15\mu+1} & \frac{12\mu-16}{15\mu+1} & \frac{21\mu+7}{15\mu+1} & \frac{-15\mu-1}{15\mu+1} \end{pmatrix} \tag{B.2}$$

and $\mathbf{U}_m\mathbf{V}_m = \mathbf{I}$. Hence

$$\mathbf{M}_m^n = \mathbf{V}_m\mathbf{D}_m^n\mathbf{U}_m. \tag{B.3}$$

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