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Bézier surfaces of minimal area: The Dirichlet approach ☆

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Abstract

The Plateau–Bézier problem consists in finding the Bézier surface with minimal area from among all Bézier surfaces with prescribed border. An approximation to the solution of the Plateau–Bézier problem is obtained by replacing the area functional with the Dirichlet functional. Some comparisons between Dirichlet extremals and Bézier surfaces obtained by the use of masks related with minimal surfaces are studied.

Keywords: Minimal surfaces; Bézier surfaces; Dirichlet functional; Permanence masks; Plateau problem

1. Introduction

The problem of finding a surface that minimizes the area with prescribed border is called the *Plateau problem*, after the Belgian researcher Plateau (the reader can see do Carmo (1976) for an informative description of the problem). Such surfaces are characterized by the fact that the mean curvature vanishes.

Statement of the Plateau problem. To find the surface of minimal area from among all surfaces with prescribed border.

In this paper we study a restricted Plateau problem: the space of possible surfaces is limited to the space of Bézier surfaces. Note that, therefore, the boundary curves must be Bézier curves.

Statement of the Plateau–Bézier problem. To find the surface of minimal area from among all Bézier surfaces with prescribed border.

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In a previous paper (Cosín and Monterde, 2002) the authors conducted a study of which surfaces with vanishing mean curvature (H) are polynomial surfaces of degree 2 and 3, i.e., they admit a Bézier form. One of the consequences of the study was to realize that such surfaces are too rigid if we want, for example, to solve blending problems. The condition $H \equiv 0$ imposes too many restrictions on a polynomial surface so that, given a prescribed border, we cannot expect to be able to find a minimal polynomial surface with that border. In fact, it can be proved that a non trivial bicubical Bézier surface is minimal if and only if it is a part of a classic example in the theory of minimal surfaces: Enneper's surface.

In general, solutions of the Plateau–Bézier problem do not need to be solutions of the Plateau (or unrestricted) problem: there could exist non-polynomial surfaces with the same border but with smaller area. In other words, a solution of the Plateau problem does not need to be a polynomial surface, even if the prescribed border is polynomial.

If tangent planes at the border are also prescribed, then the statement can be rewritten as:

Statement of the C^1-Plateau–Bézier problem. To find the surface of minimal area from among all Bézier surfaces with prescribed border and with prescribed tangent planes at the border.

When trying to solve both problems one has to minimize the area functional (see below), but this functional is highly nonlinear. This is one of the reasons that left the Plateau problem unsolved for more than a century. It was in 1931 when Douglas obtained the solution thanks to a brilliant observation (see Nitsche (1989) for a full explanation). Douglas changed the area functional to another functional, the Dirichlet one (see (1) below), which was easier to manage and has one important property: both functionals have the same extremals in the unrestricted case.

In the Bézier case this main property is no longer true in general, but what we obtain instead is that the Dirichlet extremals are an approximation to the extremals of the area functional, i.e., the resulting Bézier surface does not minimize area, but its area is close to the minimum.

There are other methods to find approximations to the solutions of the Plateau–Bézier problem. For example, in Farin and Hansford (1999) one such method is proposed. A generation scheme for the control net of a Bézier surface using a mask derived from the discretization of the Laplacian operator. The Dirichlet approach provides us with an alternative method with a similar degree of complexity. In both methods, all we have to do is to solve a system of linear equations.

Finally, it should be pointed out that in Monterde (2003) we have shown the uniqueness of the Dirichlet extremal and a convergence result: by increasing the degree, the Dirichlet extremals converge to the true minimal surface. Furthermore, in Arnal et al. (2003) we have studied the Plateau–Bézier problem for triangular patches.

Most of the plots have been colored according to the absolute value of the mean curvature of the surface.

2. The Dirichlet functional

Let $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$ be the control net of a Bézier surface and let

$$\vec{\mathbf{x}}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) P_{ij},$$

be its associated patch. The area of the Bézier surface is

$$A(\mathcal{P}) = \int_R \|\vec{\mathbf{x}}_u \wedge \vec{\mathbf{x}}_v\| \, du\, dv = \int_R \left(EG - F^2\right)^{1/2} du\, dv,$$

where $R = [0,1] \times [0,1]$ and E, F, G are the coefficients of the first fundamental form of $\vec{\mathbf{x}}$.

Since the border of a Bézier surface is determined by the boundary control points, the statement of the Plateau–Bézier problem is equivalent to the following: Given the boundary control points, $\{P_{ij}\}$ with $i = 0, n$ or $j = 0, n$, of a Bézier surface, find the inner ones in such a way that the area of the resulting Bézier surface is a minimum from among all the areas of all Bézier surfaces with the same boundary control points.

As we have said in the introduction, we do not try to minimize the area functional due to its high nonlinearity. We shall work instead with the Dirichlet functional

$$D(\mathcal{P}) = \frac{1}{2} \int_R \left(\|\vec{\mathbf{x}}_u\|^2 + \|\vec{\mathbf{x}}_v\|^2\right) du\, dv = \frac{1}{2} \int_R \big(E(u,v) + G(u,v)\big)\, du\, dv. \tag{1}$$

Such a functional was used by Douglas in order to give his famous solution to the Plateau problem. The reason is given by the following fact that relates the area and the Dirichlet functionals:

$$\left(EG - F^2\right)^{1/2} \leqslant (EG)^{1/2} \leqslant \frac{E+G}{2}. \tag{2}$$

Therefore, for any control net, $\mathcal{P}$, $A(\mathcal{P}) \leqslant D(\mathcal{P})$. Moreover, equality in (2) can occur only if $E = G$ and $F = 0$, i.e., for isothermal patches.

One difference between the two functionals is that the Dirichlet one depends on the patch. On the other hand, the area functional is independent of the patch.

Nevertheless, both functionals have a minimum in the Bézier case. First, note that they can be considered simply as functions defined on $\mathbb{R}^{3(n-1)(m-1)}$. Indeed, both functions depend on the $(n-1) \times (m-1)$ inner control points and each inner control point has three real coordinates.

Both functions are bounded from below because they are defined as integrals of positive functions. Moreover, when looking for a minimum, we can restrict both functions to a compact subset. Therefore, a classical result from calculus says that a minimum exists and it is attained.

3. Extremals of the Dirichlet functional

The next result translates the condition "a control net $\mathcal{P}$ is an extremal of the Dirichlet problem" into a system of linear equations in terms of the control points. Let us say that we are not computing the Euler–Lagrange equations of the Dirichlet functional. We will simply compute the points where the gradient of a real function defined on $\mathbb{R}^{3(n-1)(m-1)}$ vanishes.

Proposition 3.1. *A control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$0 = \frac{n^2}{2(2n-2)m} \binom{n-1}{i} \binom{m}{j} \sum_{k,\ell=0}^{n-1,m} A_{ni}^k \frac{\binom{m}{\ell}}{\binom{2m}{j+\ell}} \Delta^{10} P_{k\ell}$$

$$+\frac{m^2}{2(2m-2)n}\binom{n}{i}\binom{m-1}{j}\sum_{k,\ell=0}^{n,m-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}A^{\ell}_{mj}\Delta^{01}P_{k\ell}, \tag{3}$$

for any $i\in\{1,\dots,n-1\}$ *and* $j\in\{1,\dots,m-1\}$ *where* A^k_{ni} *is defined by*

$$\frac{ni-nk-i}{(n-i)(2n-1-i-k)}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}.$$

Proof. Let us compute the gradient of the Dirichlet functional with respect to the coordinates of a control point $P_{ij}=(x^1_{ij},x^2_{ij},x^3_{ij})$. For any $a\in\{1,2,3\}$, $i\in\{1,\dots,n-1\}$ and any $j\in\{1,\dots,m-1\}$

$$\frac{\partial D(\mathcal{P})}{\partial x^a_{ij}}=\int_R\left(\left\langle\frac{\partial\vec{\mathbf{x}}_u}{\partial x^a_{ij}},\vec{\mathbf{x}}_u\right\rangle+\left\langle\frac{\partial\vec{\mathbf{x}}_v}{\partial x^a_{ij}},\vec{\mathbf{x}}_v\right\rangle\right)du\,dv.$$

Let us compute now the partial derivatives

$$\frac{\partial\vec{\mathbf{x}}_u}{\partial x^a_{ij}}=\frac{\partial}{\partial x^a_{ij}}\frac{\partial}{\partial u}\vec{\mathbf{x}}=\frac{\partial}{\partial u}\frac{\partial}{\partial x^a_{ij}}\vec{\mathbf{x}}=\frac{\partial}{\partial u}B^n_i(u)B^m_j(v)e^a=n\big(B^{n-1}_{i-1}(u)-B^{n-1}_i(u)\big)B^m_j(v)e^a,$$

where e^a, $a\in\{1,2,3\}$, denotes the ath vector of the canonical basis, i.e., $e^1=(1,0,0)$, $e^2=(0,1,0)$, $e^3=(0,0,1)$. Analogously

$$\frac{\partial\vec{\mathbf{x}}_v}{\partial x^a_{ij}}=mB^n_i(u)\big(B^{m-1}_{j-1}(v)-B^{m-1}_j(v)\big)e^a.$$

Therefore

$$\begin{aligned}\frac{\partial D(\mathcal{P})}{\partial x^a_{ij}}&=\int_R\Big(n\big(B^{n-1}_{i-1}(u)-B^{n-1}_i(u)\big)B^m_j(v)\langle e^a,\vec{\mathbf{x}}_u\rangle\\&\qquad+mB^n_i(u)\big(B^{m-1}_{j-1}(v)-B^{m-1}_j(v)\big)\langle e^a,\vec{\mathbf{x}}_v\rangle\Big)du\,dv\\&=\int_R\left(n\big(B^{n-1}_{i-1}(u)-B^{n-1}_i(u)\big)B^m_j(v)\left\langle e^a,n\sum_{k,\ell=0}^{n-1,m}B^{n-1}_k(u)B^m_\ell(v)\Delta^{10}P_{k\ell}\right\rangle\right.\\&\qquad\left.+mB^n_i(u)\big(B^{m-1}_{j-1}(v)-B^{m-1}_j(v)\big)\left\langle e^a,m\sum_{k,\ell=0}^{n,m-1}B^n_k(u)B^{m-1}_\ell(v)\Delta^{01}P_{k\ell}\right\rangle\right)du\,dv.\end{aligned}$$

Applying now that for any $n\in\mathbb{N}$ and for any $i\in\{0,\dots,n\}$, $\int_0^1B^n_i(t)\,dt=1/(n+1)$, we get

$$\begin{aligned}\frac{\partial D(\mathcal{P})}{\partial x^a_{ij}}&=\frac{n^2}{2(2n-2)m}\sum_{k,\ell=0}^{n-1,m}\left(\frac{\binom{n-1}{i-1}\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}-\frac{\binom{n-1}{i}\binom{n-1}{k}}{\binom{2n-2}{i+k}}\right)\frac{\binom{m}{\ell}\binom{m}{j}}{\binom{2m}{j+\ell}}\big\langle e^a,\Delta^{10}P_{k\ell}\big\rangle\\&\quad+\frac{m^2}{2(2m-2)n}\sum_{k,\ell=0}^{n,m-1}\frac{\binom{n}{k}\binom{n}{i}}{\binom{2n}{i+k}}\left(\frac{\binom{m-1}{j-1}\binom{m-1}{\ell}}{\binom{2m-2}{j+\ell-1}}-\frac{\binom{m-1}{j}\binom{m-1}{\ell}}{\binom{2m-2}{j+\ell}}\right)\big\langle e^a,\Delta^{01}P_{k\ell}\big\rangle\\&=\frac{n^2}{2(2n-2)m}\binom{n-1}{i}\binom{m}{j}\sum_{k,\ell=0}^{n-1,m}A^k_{ni}\frac{\binom{m}{\ell}}{\binom{2m}{j+\ell}}\big\langle e^a,\Delta^{10}P_{k\ell}\big\rangle\end{aligned}$$

$$+\frac{m^2}{2(2m-2)n}\binom{n}{i}\binom{m-1}{j}\sum_{k,\ell=0}^{n,m-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}A_{mj}^{\ell}\langle e^a, \Delta^{01}P_{k\ell}\rangle. \qquad \square$$

In the case of a square control net, Eqs. (3) are simpler.

Corollary 3.2. *A square control net, $\mathcal{P}=\{P_{ij}\}_{i,j=0}^{n,n}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$0=\sum_{k,\ell=0}^{n-1,n}\frac{\binom{n}{\ell}}{\binom{2n}{j+\ell}}C_{ni}^{k}\Delta^{10}P_{k\ell}+\sum_{k,\ell=0}^{n,n-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}C_{mj}^{\ell}\Delta^{01}P_{k\ell}, \tag{4}$$

for any $i, j \in \{1,\dots,n-1\}$, where $C_{ni}^{k}=\frac{(n-1)i-nk}{i+k}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k}}$.

Let us recall that, as we have said in the introduction, a minimum of the Dirichlet functional with prescribed border always exists. So, fixing the boundary control points and taking the inner control points as unknowns, the linear system (3) and, in particular, the linear system (4), are always compatible and can be solved in terms of the boundary control points. See Monterde (2003) for a proof of the uniqueness of the solution.

3.1. Examples

If $n=m=2$, then there is just one equation corresponding to the inner control point P_{11}.

Proposition 3.3. *A biquadratic Bézier surface is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{11}=\frac{1}{8}(3P_{00}-P_{01}+3P_{02}-P_{10}-P_{12}+3P_{20}-P_{21}+3P_{22}). \tag{5}$$

If $n=m=3$, there are four equations corresponding to the inner control points $P_{11}, P_{12}, P_{21}, P_{22}$.

Proposition 3.4. *A bicubic Bézier surface is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{11}=\frac{1}{78}(48P_{00}-22P_{01}+24P_{02}-22P_{10}+15P_{13}+24P_{20}-4P_{23}+15P_{31}-4P_{32}+4P_{33}),$$
$$P_{12}=\frac{1}{78}(24P_{01}-22P_{02}+48P_{03}+15P_{10}-22P_{13}-4P_{20}+24P_{23}+4P_{30}-4P_{31}+15P_{32}),$$
$$P_{21}=\frac{1}{78}(15P_{01}-4P_{02}+4P_{03}+24P_{10}-4P_{13}-22P_{20}-15P_{23}+48P_{30}-22P_{31}+24P_{32}),$$
$$P_{22}=\frac{1}{78}(4P_{00}-4P_{01}+15P_{02}-4P_{10}+24P_{13}+15P_{20}-22P_{23}+24P_{31}-22P_{32}+48P_{33}).$$

3.2. Relation with harmonic patches

The Dirichlet functional can be defined as we did before (1), for just Bézier (or polynomial) patches $\vec{\mathbf{x}}^{\mathcal{P}}:[0,1]\times[0,1]\to\mathbb{R}^3$, $\vec{\mathbf{x}}^{\mathcal{P}}$ being a Bézier patch associated to a control net $\mathcal{P}$.

Or it can also be defined for arbitrary patches $\vec{\mathbf{x}}:[0,1]\times[0,1]\to\mathbb{R}^3$, $\vec{\mathbf{x}}$ being a differentiable patch.

$$D(\vec{\mathbf{x}})=\frac{1}{2}\int_R\left(\|\vec{\mathbf{x}}_u\|^2+\|\vec{\mathbf{x}}_v\|^2\right)du\,dv.$$

Let us call this case the 'unrestricted case' in contrast to the Bézier or restricted case.

In the unrestricted case, the extremals of the Dirichlet functional are given by differentiable patches verifying its Euler–Lagrange equation, $\Delta\vec{\mathbf{x}}=0$, i.e. by harmonic patches. But even when the boundary conditions are polynomial curves, the Dirichlet extremal for the unrestricted case does not need to be polynomial in general and so, it cannot be an extremal in the restricted case. Let us denote the extremal of the Dirichlet functional in the unrestricted case by $\vec{\mathbf{x}}^{\text{ext}}$, the control net extremal of the Dirichlet functional in the restricted case by $\mathcal{P}^{\text{ext}}$, and its associated Bézier patch by $\vec{\mathbf{x}}^{\mathcal{P}^{\text{ext}}}$. What we have is the following inequality

$$D(\vec{\mathbf{x}}^{\text{ext}})\leqslant D(\vec{\mathbf{x}}^{\mathcal{P}^{\text{ext}}})=D(\mathcal{P}^{\text{ext}}).$$

In general $D(\vec{\mathbf{x}}^{\text{ext}})<D(\vec{\mathbf{x}}^{\mathcal{P}^{\text{ext}}})$. Nevertheless, if a polynomial patch is harmonic, then it is an extremal of the Dirichlet functional both in the unrestricted and the restricted case.

Theorem 3.5. *Let $\mathcal{P}=\{P_{ij}\}_{i,j=0}^{n,m}$ be the control net of a Bézier surface. If the associated Bézier patch $\vec{\mathbf{x}}$ is harmonic, then it is an extremal of the Dirichlet functional from among all the Bézier patches with the same boundary.*

Yet, not all extremal patches of the Dirichlet functional in the restricted case are harmonic patches. In Cosín and Monterde (2002) we gave the conditions that a control net must satisfy for the associated Bézier surface to be harmonic. Such conditions involve not only the inner control points but also some boundary control points. For example, in a bicubic harmonic patch only two border lines of control points, eight points in all, are free. Given the first and last rows of control points, the other two rows are totally determined. In particular, there are four boundary control points that are linearly dependent on the

$$\begin{aligned}
P_{10}&=\tfrac{1}{3}(4P_{00}-4P_{01}+2P_{02}+2P_{30}-2P_{31}+P_{32}),\\
P_{20}&=\tfrac{1}{3}(2P_{00}-2P_{01}+P_{02}+4P_{30}-4P_{31}+2P_{32}),\\
P_{13}&=\tfrac{1}{3}(2P_{01}-4P_{02}+4P_{03}+P_{31}-2P_{32}+2P_{33}),\\
P_{23}&=\tfrac{1}{3}(P_{01}-2P_{02}+2P_{03}+2P_{31}-4P_{32}+4P_{33}).
\end{aligned}$$

$$\begin{matrix}P_{00}&P_{01}&P_{02}&P_{03}\\ *&*&*&*\\ *&*&*&*\\ P_{30}&P_{31}&P_{32}&P_{33}\end{matrix}\qquad\qquad\begin{matrix}P_{00}&P_{01}&P_{02}&P_{03}\\ P_{10}&*&*&P_{13}\\ P_{20}&*&*&P_{23}\\ P_{30}&P_{31}&P_{32}&P_{33}\end{matrix}$$

Fig. 1. Configuration of the boundary conditions for $n=m=3$ of the control net of a Bézier surface. Left: The harmonic case. Right: The Plateau–Bézier case.

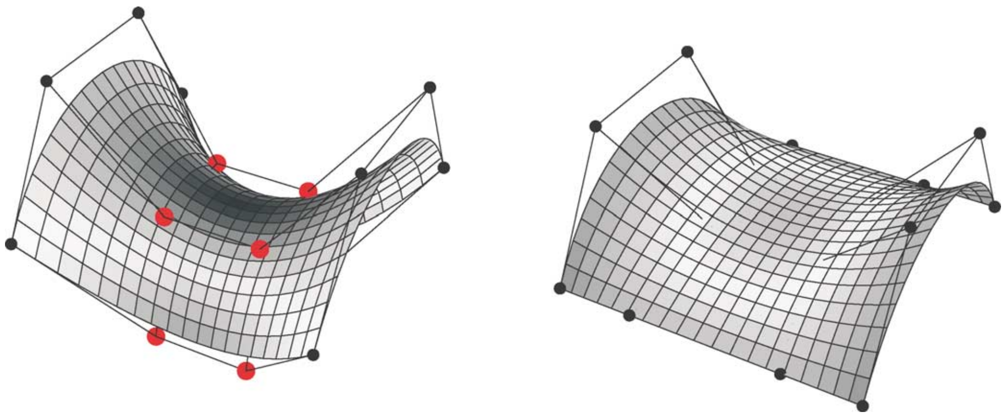

Fig. 2. Left: Harmonic Bézier surface. Small black dots are the fixed control points. Bigger dots are the computed ones. Right: Extremal of the Dirichlet functional, but not harmonic. Small black dots are fixed by the prescribed border. There are only four computed inner control points, hidden by the surface. Gray levels on the surface correspond to the absolute values of the mean curvature. White levels correspond to values close to zero.

other eight (see Fig. 1). (The general result for any degree can be seen in Monterde (2003).) The explicit relations for those boundary control points are given in Fig 1.

Only those configurations of the boundary control points that verify such relations can produce extremals of the Dirichlet functional of the restricted case which are harmonic, i.e., extremals of the Dirichlet functional in the unrestricted case. (See Fig. 2, left.)

4. Permanence patches related to the Plateau–Bézier problem

As is well known, if the boundary curves of a Bézier surface are prescribed, then the boundary control points are fixed. Therefore, the problem of constructing a Bézier surface with prescribed border consists in computing the inner control points. A simple way of constructing Bézier surfaces with prescribed boundary consists in generating the inner control points by using a mask.

Let us recall that a mask is a linear relation between one inner control point and its eight neighboring control points. What one has to do is to just solve a system of linear equations whose matrix of coefficients is a sparse matrix, i.e., a matrix with just a few non-vanishing entries. For an $n \times m$ Bézier surface, there are $(n-1) \times (m-1)$ linear equations and the same number of inner control points.

In Farin and Hansford (1999), the authors define the notion of permanence patches as being those generated by masks with the following form

$$\begin{matrix} \alpha & \beta & \alpha \\ \beta & \bullet & \beta \\ \alpha & \beta & \alpha \end{matrix} \tag{6}$$

with $4\alpha + 4\beta = 1$ (i.e., $\beta = 1/4 - \alpha$). Let us denote this mask by M_α.

They are called permanence patches because the case $\alpha = -0.25$ gives the control net generation scheme used to generate Coons patches and these Coons patches satisfy the permanence principle (see Farin and Hansford (1999)): let two points (u_0, v_0) and (u_1, v_1) define a rectangle R in the domain U of the Coons patch. The four boundaries of this subpatch will map to four curves on the Coons patches. The Coons patch for those four boundary curves is the original Coons patch, restricted to the rectangle R.

Moreover, as it is also explicitly said in the same reference, all schemes whose construction satisfies a variational principle share this permanence property.

One of the cases studied therein is the mask $\alpha = 0$ corresponding to a discretization of the Laplacian operator. As surfaces of minimal area are related to harmonic patches (you can see any book on classical differential geometry, for example do Carmo (1976), Gray (1998), or Osserman (1986)) then the solutions of the linear systems generated by the mask $\alpha = 0$ are an approximation to Bézier surfaces of minimal area.

Nevertheless, we can generate other masks by applying different guiding principles also related with surfaces of minimal area and, obviously, related with a variational principle. Note that surfaces of minimal area also verify a permanence principle: if we consider the boundary, B, of a subset of a given minimal surface, then the surface of minimal area from among all surfaces with the same boundary B is the original minimal surface.

4.1. The discrete Laplacian mask

It can be found in Farin and Hansford (1999) that the mask M_0 is the discrete form of the Laplacian operator. Such a mask is used in the cited reference to obtain control nets resembling minimal surfaces that fit between given boundary polygons.

The deduction of such a mask is a very well-known process coming from numerical integration by the finite difference method of partial differential equations. Transferring the second order central difference approximation of a differentiable function to the control net of a Bézier surface, we obtain the following formula

$$P_{ij} = \frac{1}{4}(P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1}),$$

and this formula corresponds to the mask $\alpha = 0$.

Note that $M_0(P_{ij})$ is the center of gravity of the four neighboring points of P_{ij}, which are not at the corners.

Should also be noted that what we really obtain is an approximation of a *harmonic* control net, but not, in principle, an approximation of a harmonic Bézier patch. This will be evident later when comparing different masks on a biquadratic Bézier surface. Nevertheless, for higher degree Bézier surfaces and when the boundary control points are close to the boundary curves, the control net is indeed an approximation to the Bézier surface.

4.2. The harmonic mask

Instead of discretizing the Laplacian operator, let us demand that, at least at one point, the Laplacian of the patch vanishes. So, we are not doing an approximation to a *harmonic* control net. What we are trying to do is to transfer the harmonic condition of the patch into a condition on the control net.

Proposition 4.1. *The Bézier patch, $\vec{\mathbf{x}}$, associated to a biquadratic control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{2,2}$, verifies $\Delta\vec{\mathbf{x}}(\frac{1}{2}, \frac{1}{2}) = 0$ if and only if*

$$P_{11} = M_{1/4}(P_{11}). \tag{7}$$

Proof. If $\vec{\mathbf{x}}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}B_i^2(u)B_j^2(v)P_{ij}$ then

$$\begin{aligned}\Delta\vec{\mathbf{x}}(u,v)&=2\sum_{j=0}^{2}B_0^0(u)B_j^2(v)\Delta^{20}P_{0j}+2\sum_{i=0}^{2}B_i^2(u)B_0^0(v)\Delta^{02}P_{i0}\\&=2\sum_{j=0}^{2}B_j^2(v)\Delta^{20}P_{0j}+2\sum_{i=0}^{2}B_i^2(u)\Delta^{02}P_{i0}.\end{aligned}$$

Therefore, a single computation shows that

$$\Delta\vec{\mathbf{x}}\left(\frac{1}{2},\frac{1}{2}\right)=P_{20}+P_{00}-4P_{11}+P_{22}+P_{02}.\qquad\square$$

This mask was also obtained in Cosín and Monterde (2002) as one of the conditions that a biquadratic patch must satisfy in order to be globally harmonic.

Finally, note that, conversely to what happens for the mask M_0, now $M_{1/4}(P_{ij})$ is the center of gravity of the four neighboring points at the corners.

4.3. The Dirichlet mask

The third mask is given by the Dirichlet equations for $n=m=2$. Rewriting Proposition 3.3 in terms of masks, we now have that

Proposition 4.2. *A biquadratic control net, $\mathcal{P}=\{P_{ij}\}_{i,j=0}^{2,2}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{11}=M_{3/8}(P_{11}).\tag{8}$$

The Dirichlet mask corresponds to the value $\alpha=3/8$. We can write $M_{3/8}$ as $3/2M_{1/4}-1/2M_0$. Therefore, $M_{3/8}(P_{ij})$ is a linear combination between the centers of gravity of the four neighboring corner points and the other four neighboring points that are not at the corners.

5. Comparison between the three masks

The obvious question then, is to determine which one is the best, or even more generally, whether there is or not a better mask. The answer is negative. The highly nonlinearity of the area functional makes the dependence of the minimal surface from the boundary conditions highly nonlinear too. So, one cannot expect a mask, i.e., a linear expression, to be able to give a good approximation in all cases. This is true for the minimal case, but for any case the situation is rather similar. As has already been said in Farin and Hansford (1999) "... a single choice of α will not produce good shape. The appropriate value depends on the geometry of the boundary curves."

We will show some examples with simple boundary curves and where the Dirichlet mask is better than the other two: it provides Bézier surfaces with smaller area than in the case of the other two masks. But it is also easy to provide examples showing the opposite behaviour.

5.1. Case $n = m = 2$

Let us start the comparison by studying some examples in the biquadratic case.

For an arbitrary mask, the only inner point is given by

$$P_{11}^{\alpha} = \alpha(P_{00} + P_{02} + P_{20} + P_{22}) + \left(\frac{1}{4} - \alpha\right)(P_{01} + P_{10} + P_{12} + P_{21}).$$

Given fixed boundary control points, let $\vec{\mathbf{x}}^{\alpha}$ (respectively, $\mathcal{P}^{\alpha}$) be the associated Bézier surface (respectively, control net). In this simple case, we can explicitly compute the areas, $A(\mathcal{P}^{\alpha})$, of $\vec{\mathbf{x}}^{\alpha}$ for any real number α. The value $\alpha_{\min}$, for which the function $A(\mathcal{P}^{\alpha})$ has a minimum, provides the solution to the Plateau–Bézier problem.

Fig. 3 shows an example of boundary conditions and the three Bézier surfaces obtained by the different masks. In this example the approximation given by the Dirichlet mask is better that the other two masks. The resulting areas are 72.8080 ($\alpha = 0$), 67.7838 ($\alpha = 0.25$) and 67.1954 ($\alpha = 0.375$). Moreover, the Dirichlet extremal is very near to the true minimal surface $\alpha \sim 0.3675$ and a minimal area 67.1929, i.e., a difference between areas of 0.004%.

Similar studies can be made for different configurations of the boundary conditions. If the boundary conditions are not too strange, then the behaviour of the area function is similar and the Dirichlet mask is the best one. Nevertheless, there is one interesting case. Let us recall that the mask M_0 computes the center of gravity of four of the neighboring points, whereas $M_{1/4}$ computes the center of gravity of the other four.

For an arbitrary mask M_{α}, let us express P_{11} as follows:

$$\begin{aligned} P_{11}^{\alpha} = M_{\alpha}(P_{11}) &= \alpha\big((P_{00} + P_{02} + P_{20} + P_{22}) - (P_{01} + P_{10} + P_{12} + P_{21})\big) \\ &\quad + \frac{1}{4}(P_{01} + P_{10} + P_{12} + P_{21}) \end{aligned}$$

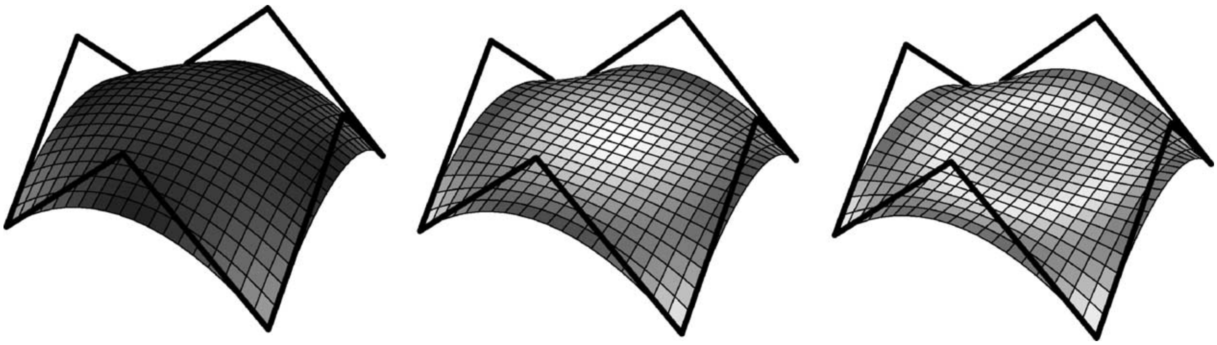

Fig. 3. The $n = m = 2$ Bézier surfaces associated to the same boundary conditions but with different masks. Left, the discretization of the Laplacian operator ($\alpha = 0$). Center, the harmonic mask ($\alpha = 0.25$). Right, the Dirichlet mask ($\alpha = 0.375$). Control points are located on circles of radius 4. Gray levels on the surfaces correspond to the absolute values of the mean curvature. White levels are related to values close to zero. Note how, for the best result, the white zones are located in a centered ring. It must be remembered that we cannot expect to obtain a totally white surface because there is not a minimal ($H \equiv 0$) Bézier surface for this boundary configuration. It can be shown that minimal polynomial surfaces of degree 2 are pieces of planes. As the border configuration of this example is not planar, then there is a minimal Bézier surface. In fact, there is a minimal surface with the same border, but it is not a polynomial one.

$$= \alpha\big(M_{1/4}(P_{11}) - M_0(P_{11})\big) + \frac{1}{4}M_0(P_{11}).$$

So, if the configuration of the boundary points of a biquadratic control net is such that both centers of gravity are located at the same point, then the central point P_{11} does not depend on α. Therefore, for such a configuration of the boundary, any mask will define the same Bézier surface.

5.2. Case $n = m = 4$

The boundary conditions we shall study and their associated Bézier surfaces for the Dirichlet mask are shown in the figures below (Fig. 4).

The areas of the Bézier surfaces in Fig. 4 are shown in Table 1.

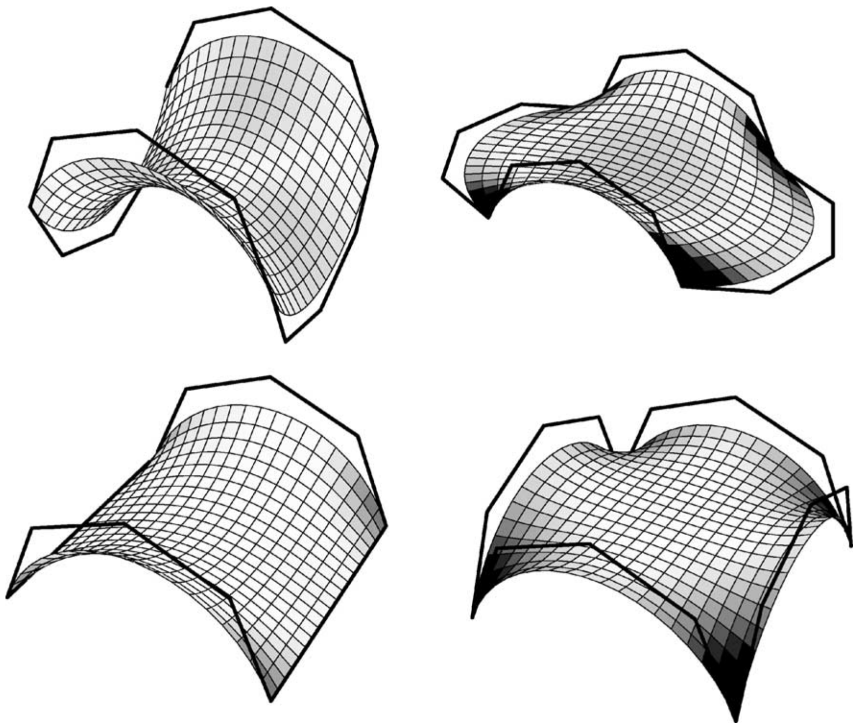

Fig. 4. Some different boundary conditions and the associated Bézier surfaces. Control points not lying on a straight line are located on circles with radius 4. The drawn surfaces have been obtained with the Dirichlet mask.

Table 1
Comparison between the areas of the surfaces shown in Fig. 4 obtained by different methods

Mask	Top left	Top right	Bottom left	Bottom right
$\alpha = 0$	101.356 (99.91%)	109.316 (101.13%)	77.3515 (100, 91%)	71.3129 (102.56%)
$\alpha = \frac{1}{4}$	101.432 (99.98%)	108.849 (100.70%)	76.9206 (100.35%)	69.6413 (100.16%)
$\alpha = \frac{3}{8}$	101.457 (100.01%)	108.762 (100.62%)	76.8465 (100.25%)	69.4261 (99.85%)
Dirichlet extremal	101.449 (100.00%)	108.094 (100.00%)	76.6552 (100.00%)	69.5302 (100.00%)

In the last three examples the best mask is always the Dirichlet mask, whereas, in the first case, the best one is M_0. This is another example of how the solutions depend heavily on the boundary conditions. In the table we have added a last row with the results for the Dirichlet extremal. The numbers in parentheses refer to the percent by which the area differs from the area of the Dirichlet extremal.

For these configurations at least, the Dirichlet extremal always improves the results obtained by the Dirichlet mask. Only in the top left case are the results of the other two masks better than the Dirichlet extremal. A possible explanation of this fact will be dealt with in the next section, but before that it is interesting to note (and we thank the referee for pointing out this fact) that the area of the translational surface (see Farin (2001)) that has the same boundary as in the top left case is 101.349, i.e., smaller than the other four approximations.

In the bottom cases, which are the other two where a translational surface with the same boundary can be defined, the shape of the translational surfaces shows that they are very far from being of minimal area.

5.3. Higher degree examples

Let us see what happens in the following two examples (Fig. 5) with $n = m = 8$. In the first case we have tried the same boundary conditions as in Farin and Hansford (1999) (Fig. 3 therein). In the second case, we have changed the boundary curves a little.

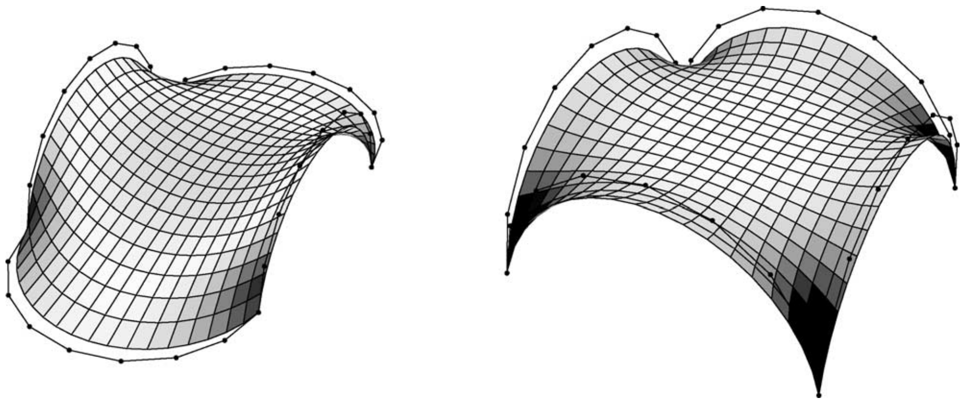

Fig. 5. Boundary control points and Bézier surfaces generated by the mask $\alpha = 0.375$. Case I, left, the same boundary conditions as in Farin and Hansford (1999). Case II, right, two of the boundary curves have been changed. Again, the gray levels correspond to the values of the mean curvature on a scale common to both surfaces.

Table 2
Comparison between the areas of the surfaces shown in Fig. 5

Mask	Area case I	Area case II
$\alpha = 0$	120.262 (100.59%)	70.0807 (98.33%)
$\alpha = 0.25$	120.134 (100.49%)	70.0667 (98.31%)
$\alpha = 0.375$	120.103 (100.46%)	70.0878 (98.34%)
Dirichlet extremal	119.551 (100.00%)	71.2706 (100.00%)

Finally, let us now compare the area of the Bézier surfaces in Fig. 5 associated to the three control nets.

Again, the numbers in parentheses refer to the percent by which the area differs from the area of the Dirichlet extremal. The best mask in the first case is again the Dirichlet one. But in the second case the Dirichlet mask is the worst.

A possible explanation of why the Dirichlet extremals (and the Dirichlet mask) fail in some cases could be the following: It should be remembered that the coefficients of the first fundamental form at the vertex of a Bézier surface only depend on the boundary control points. In both cases we have $E = G = 155.895$. But the first configuration can be considered as more isothermal than the other because, in the first case $F = -29.8292$ whereas in the second case, $F = 149.96$. The angle at the corners in the first case ($\approx 106^\circ$) is nearly a right angle, whereas in the second case the angle ($\approx 15^\circ$) is far from being a right angle.

Also note that gray levels close to black in both surfaces of Fig. 5 indicate points with mean curvature comparatively higher than zero, and that these points are located at the corners. The darkest zones in the figure on the right are wider than in the one on the left.

Note that in this second case, the inequalities of (2) are far from being equalities near the corners of the Bézier surfaces. Any Bézier patch with such boundary conditions will always fail to be isothermal at the corners. So, any approximation based on the substitution of the area functional by the Dirichlet one will have an intrinsic error due to the method.

5.4. Rectangular case

Let us have a look at the behavior of the masks and the Dirichlet extremals for rectangular Bézier surfaces.

The areas of the Bézier surfaces in Fig. 6 are shown in Table 3.

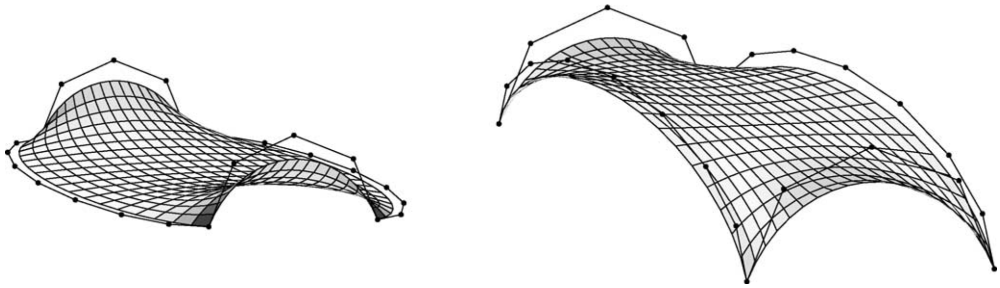

Fig. 6. Rectangular boundary control points and Bézier surfaces generated by the mask $\alpha = 0.375$. Case I, left, the same boundary conditions as in Farin and Hansford (1999). Case II, right, two of the boundary curves have been changed.

Table 3
Comparison between the areas of the surfaces shown in Fig. 6

Mask	Area case I	Area case II
$\alpha = 0$	435.237 (98.66%)	265.211 (98.42%)
$\alpha = 0.25$	434.318 (98.45%)	264.398 (98.12%)
$\alpha = 0.375$	434.132 (98.41%)	264.303 (98.08%)
Dirichlet extremal	441.162 (100.00%)	269.470 (100.00%)

Now, at the corners, for case I,

$$E = 244.896, \qquad G = 149.961, \qquad F = -57.388,$$

and for case II,

$$E = 244.896, \qquad G = 149.961, \qquad F = 138.546.$$

Also note, that in this rectangular case, the Dirichlet extremal is a worse approximation than the ones obtained with any of the three masks. In fact, things run better now for the Dirichlet mask.

6. Comparison between masks and Dirichlet extremals

With the same boundary conditions as in Farin and Hansford (1999) (Fig. 3 therein), the control net defined by the Dirichlet masks is very similar to that of the cited reference (see Fig. 7 below).

If we now apply our results for the same boundary curves with $n = m = 8$ what we obtain is a control net extremal of the Dirichlet functional, i.e., its associated Bézier surface (Fig. 7) minimizes the sum of $\|\vec{\mathbf{x}}_u\|^2 + \|\vec{\mathbf{x}}_v\|^2$. Note that the number of linear equations are the same as in the previous case.

The control net generating such a surface is not so pleasant as the control net shown in Farin and Hansford (1999) or in Fig. 7. Its complexity prevents us from drawing it completely. In Fig. 7 just two lines of control points are pictured.

The difference between the Dirichlet extremal control net (Fig. 8) and the control net in Fig. 3 in Farin and Hansford (1999) can be explained as follows: In both cases the key point is to use a variational principle. In our approach we are looking for Bézier surfaces minimizing some functional, so the main object is the surface, not its control net. In the approach used in Farin and Hansford (1999), the authors are looking for control nets that verify some discrete version of a condition coming from a variational principle. So, in that approach, the main object is not the Bézier surface, but its control net.

The Dirichlet extremals shown in Fig. 5 are examples of Bézier surfaces with disorderly associated control net (see Fig. 8) but, in one of the cases, with less area than other approaches that focus on the control net.

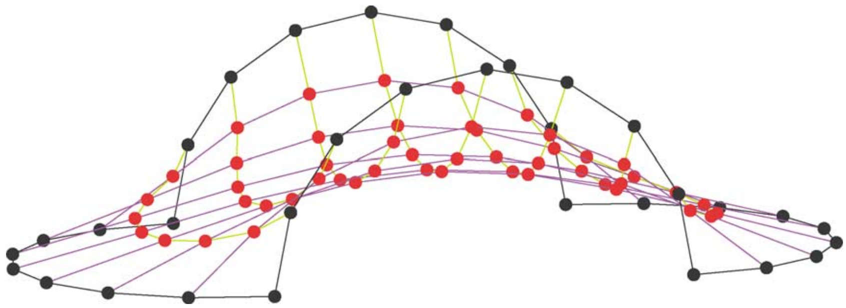

Fig. 7. Control net of the permanence patch defined by the mask $\alpha = 3/8$.

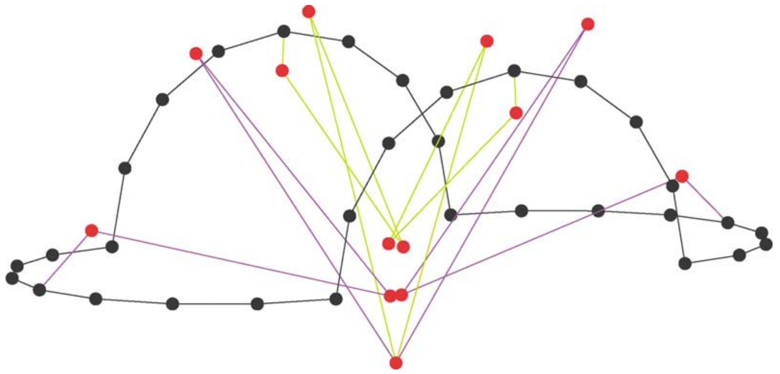

Fig. 8. Some of the inner control points of the Bézier surface shown in Fig. 5, left.

7. Improvement of the approximation

As we have seen in the previous examples, the Dirichlet extremal is a good approximation to the solution of the Plateau–Bézier problem only when the first fundamental form of the surface at the corners is close to isothermality. The non-isothermality at corner points produces an error that is intrinsic to the method when substituting the area functional by the Dirichlet one. At points other than the corner points, the configuration of the Dirichlet extremal tends to the isothermality of the patch. But at the corner points, the first fundamental form is fixed from the border control points and it cannot be modified. This is why the Dirichlet extremal does not improve the results obtained using a mask in some cases.

Throughout this section we will propose a new method also based on differential geometric arguments but which increases the computational cost. Using the Dirichlet extremal as a first approximation to the solution of the Plateau–Bézier problem, we will find a new and better approximation. This new approximation is computed thanks to a system of linear equations, as before, but now the coefficients of the system are the result of a set of integrals of some functions that depend on the previous approximation.

The new method is based on the following result:

Proposition 7.1. *A patch $\vec{\mathbf{x}}$ is minimal iff $\Delta^g \vec{\mathbf{x}} = 0$ where g represents the first fundamental form of $\vec{\mathbf{x}}$ and Δ^g is the associated Laplacian operator*: *for a function f*:

$$\Delta^g f = \left(\frac{f_u G - f_v F}{\sqrt{EG - F^2}} \right)_u + \left(\frac{-f_u F + f_v E}{\sqrt{EG - F^2}} \right)_v .$$

Note that when the patch is isothermal, then Δ^g is nothing but the usual Laplacian operator.

It is easy to check that, for a given metric, g, with coefficients E, F and G, the equation $\Delta^g \vec{\mathbf{x}} = 0$ is the Euler–Lagrange equation of the functional

$$\mathcal{D}^g(\vec{\mathbf{x}}) = \int_{\mathcal{R}} \left(\frac{\|\vec{\mathbf{x}}_u\|^2 G - 2\langle \vec{\mathbf{x}}_u, \vec{\mathbf{x}}_v \rangle F + \|\vec{\mathbf{x}}_v\|^2 E}{\sqrt{EG - F^2}} \right) du\, dv = \int_{\mathcal{R}} g^{-1}(d\vec{\mathbf{x}}, d\vec{\mathbf{x}}) \mu_g,$$

where $\mu_g = \sqrt{EG - F^2}\, du\, dv$ is the metric volume element.

If $\mathcal{P}$ is a control net, then $\mathcal{D}^g(\mathcal{P}) := \mathcal{D}^g(\vec{\mathbf{x}}^{\mathcal{P}})$ where $\vec{\mathbf{x}}^{\mathcal{P}}$ denotes the Bézier patch associated to the control net.

Note that for a given g, the extremal of the functional $\mathcal{D}^g$ is a control net that can be computed thanks to a linear system. Therefore, the correction of the Dirichlet method is the following: let $\vec{\mathbf{x}}_0$ be the patch associated to the Dirichlet extremal and let g_0 be its first fundamental form. The new approximation is the extremal of the functional $\mathcal{D}^{g_0}$, that is, using the Dirichlet extremal as the fixed metric. Note that the functional $\vec{\mathbf{x}} \to \mathcal{D}^{g_0}(\vec{\mathbf{x}})$ is quadratic in $\vec{\mathbf{x}}$. Therefore the extremal equations are linear.

In order to state the next result, we need to define some functions. For all $i, k \in \{1, 2, \dots, n-1\}$ and $j, \ell \in \{1, 2, \dots, m-1\}$

$$\begin{aligned}
A_{ijk\ell}(u, v) &= n^2\big(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\big)B_j^m(v)\big(B_{k-1}^{m-1}(u) - B_k^{m-1}(u)\big)B_\ell^m(v),\\
B_{ijk\ell}(u, v) &= nm\big(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\big)B_j^m(v)B_\ell^n(v)\big(B_{\ell-1}^{m-1}(u) - B_\ell^{m-1}(u)\big),\\
C_{ijk\ell}(u, v) &= nmB_i^n(v)\big(B_{j-1}^{m-1}(u) - B_j^{m-1}(u)\big)\big(B_{k-1}^{n-1}(u) - B_k^{n-1}(u)\big)B_\ell^m(v),\\
D_{ijk\ell}(u, v) &= m^2B_i^n(v)\big(B_{j-1}^{m-1}(u) - B_j^{m-1}(u)\big)B_\ell^n(v)\big(B_{\ell-1}^{m-1}(u) - B_\ell^{m-1}(u)\big).
\end{aligned} \tag{9}$$

Finally, let

$$M_{ij}^{k,\ell} = \int\limits_R \frac{A_{k\ell ij}G_0 - (B_{k\ell ij} + C_{k\ell ij})F_0 + D_{k\ell ij}E_0}{\sqrt{EG - F^2}}\,du\,dv.$$

Proposition 7.2. *Let $\vec{\mathbf{x}}_0$ be a Bézier patch with prescribed border and let g_0 denote its first fundamental form with coefficients E_0, F_0 and G_0. A control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$, is an extremal of the functional $\mathcal{D}^g$ with prescribed border if and only if*

$$\sum_{k,\ell=1}^{n-1,m-1} M_{ij}^{k\ell}P_{k\ell} = -\sum_{P_{k\ell}\ \text{boundary control point}} M_{ij}^{k\ell}P_{k\ell}$$

for all $i \in \{1, 2, \dots, n-1\}$ and $j \in \{1, 2, \dots, m-1\}$.

The proof is similar to that of Proposition 3.1.

The formulas obtained in Proposition 7.2 give us a system of linear equations for the interior points of the quadrangular net given its border.

Now, if we have a look at Table 4, we can see that this method improves the results obtained using all the other methods and, moreover, we get this improvement for all the examples, even when we deal with non-isothermal charts.

The main drawback of this method of improvement is the computation of the integrals $M_{ij}^{k\ell}$.

Table 4
Improvement of the results of Table 1 by using the Dirichlet extremal as initial approximation

	Top left	Top right	Bottom left	Bottom right
Area of the new extremal	101.302	107.486	76.6509	69.1658

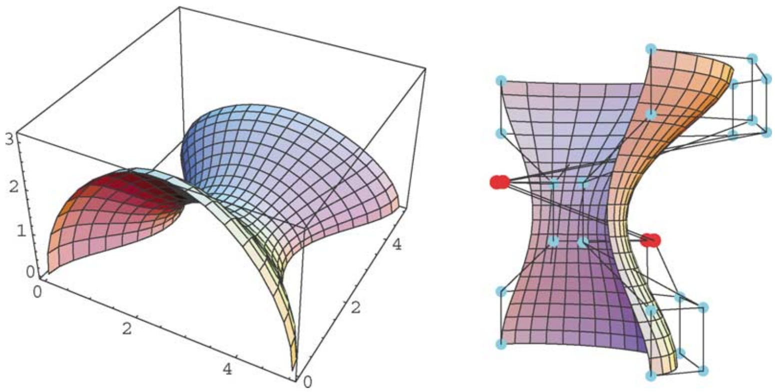

Fig. 9. Two views of the same surface: the extremal of the Dirichlet functional with prescribed border and tangent planes at the border.

8. The C^1-Plateau–Bézier problem

It is well known that tangent planes at the border of a Bézier surface are determined by the two boundary lines of control points. The statement of the C^1-Plateau–Bézier problem is now equivalent to the following one: "*Given the two lines of boundary control points,* $\{P_{ij}\}$ *with* $i \in \{0, 1, n-1, n\}$ *or* $j \in \{0, 1, m-1, m\}$*, of a Bézier surface, find the inner ones in such a way that the area of the resulting Bézier surface is a minimum from among all the areas of all Bézier surfaces with the same boundary control points.*"

We can study the linear system consisting in the same equations (3) as in Proposition 3.1 but just for $i = 2, \ldots, n-2$ and $j = 2, \ldots, m-2$ with the inner control points $\{P_{ij}\}_{i,j=2}^{n-2,m-2}$ as unknowns.

In the case $n = m = 5$ there are four equations corresponding to the inner control points $P_{22}, P_{23}, P_{32}, P_{33}$. The resulting system can be easily solved but we do not include it here due to the complexity of the expressions involved in the solution. We present instead an example in Fig. 9.

9. C^1-masks for Bézier surfaces of minimal area

As in the C^0-case, the system of linear equations to be solved to find the Dirichlet extremal has a matrix without null entries. It would be better to work with a sparse matrix like the ones that appear when working with masks. This can be achieved with the use of C^1-masks.

We can extend the defining principles of the masks set out above to the C^1 case.

Proposition 9.1 (The harmonic C^1-mask). *The Bézier patch,* $\vec{\mathbf{x}}$*, associated to a* $n = m = 4$ *control net,* $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{4,4}$*, verifies* $\Delta\vec{\mathbf{x}}(\frac{1}{2}, \frac{1}{2}) = 0$ *if and only if* $P_{22} = M(P_{22})$*, where* M *is the* C^1*-mask*

$$\frac{1}{12}\begin{array}{ccccc} 1 & 2 & 2 & 2 & 1 \\ 2 & 0 & -4 & 0 & 2 \\ 2 & -4 & \bullet & -4 & 2 \\ 2 & 0 & -4 & 0 & 2 \\ 1 & 2 & 2 & 2 & 1 \end{array} \tag{10}$$

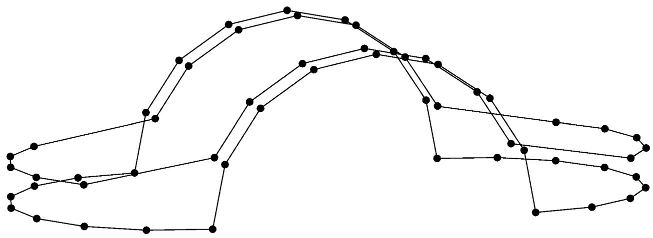

Fig. 10. C^1 boundary conditions for Fig. 11.

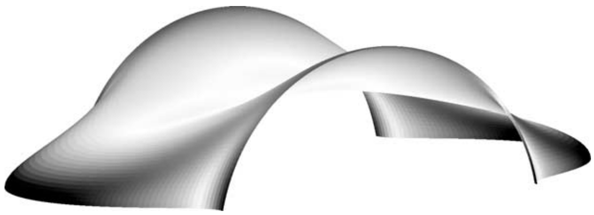

Fig. 11. Another example of a Dirichlet extremal for a C^1-Plateau–Bézier problem. The C^1-boundary conditions are given in Fig. 10. Note how the gray zones are now located along the whole boundary curves and not just at the corner points.

Proposition 9.2 (The Dirichlet C^1-mask). *The Bézier patch, $\vec{\mathbf{x}}$, associated to a $n = m = 4$ control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{4,4}$, is an extremal of the Dirichlet functional if and only if $P_{22} = M(P_{22})$, where M is the C^1-mask*

$$\frac{1}{48}\begin{matrix} 20 & 15 & 14 & 15 & 20 \\ 15 & -20 & -32 & -20 & 15 \\ 14 & -32 & \bullet & -32 & 14 \\ 15 & -20 & -32 & -20 & 15 \\ 20 & 15 & 14 & 15 & 20 \end{matrix} \tag{11}$$

Fig. 11 is another example of a Dirichlet extremal for a C^1-Plateau–Bézier problem. The area of the Bézier surface obtained with the harmonic C^1-mask is 124.040; with the Dirichlet C^1-mask the area is 123.502, whereas the area of the Dirichlet extremal is better: 121.371.

10. Conclusion

The high nonlinearity of the area functional made it extremely difficult to work with. Borrowing an argument from the theory of minimal surfaces, the area functional is substituted by the Dirichlet functional. Now, the extremals of such a functional can be easily computed as the solutions of linear systems. They are not extremals of the area functional but they are a fine approximation in some cases.

Some authors (Farin and Hansford, 1999) have proposed a way of obtaining approximations to minimal surfaces with prescribed boundary curves by using a mask. The computation of the Dirichlet extremals is an alternative way of finding such approximations but with an increase in the computational cost because, although both methods are based on the resolution of a system of linear equations of the same size, with the use of masks the matrix of coefficients is a sparse matrix, whereas in the Dirichlet case, the matrix of coefficients has no zeros.

We propose two new masks related to the Plateau–Bézier problem. One is related to the Laplacian operator, and the second is associated to the Dirichlet approach. A comparison between the results of the three masks and the Dirichlet extremals for several different configurations of the boundary conditions has been performed.

There is no best choice, but the examples and theoretical arguments point out that when the first fundamental form of the Bézier surface at the corners (at these points the IFF depends on just the boundary conditions) is close to being isothermal, then the Dirichlet extremal is a better approximation than the ones obtained by the use of masks.

On the other hand, if the first fundamental form of the Bézier surface at the corners is far from being isothermal, then the results obtained by the use of a mask can be better than the result obtained by the Dirichlet extremal.

Some authors (Greiner, 1994; Moreton and Séquin, 2001) have proposed iterative methods aimed at reaching a minimum of some functionals related with area or with the mean curvature. The extremals of the Dirichlet functional are an alternative way of obtaining, without integration, an approximation of the surface minimizing area. In any case, if one wants to obtain better approximations, the extremals of the Dirichlet functionals can be used as the starting point for recursive algorithms that optimize the area functional.

Without going into an iterative method, we propose an improvement of the Dirichlet method that gives better results than the previous one, but which now has a really high computational cost.

If one still wants to maintain the use of masks instead of Dirichlet extremals, then we propose a distinguished mask, the Dirichlet mask, corresponding to $\alpha = 3/8$. Experimental results for rectangular control nets show that even when the Dirichlet extremal does not work very well, the results obtained by the use of the Dirichlet mask are better than those obtained by the other two distinguished masks.

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