Geometry of Minkowski Space

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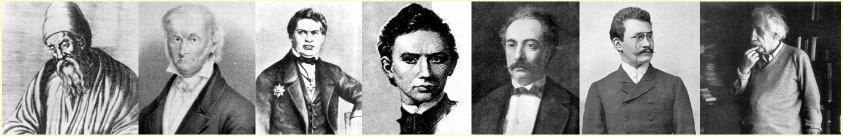
Habilitation lecture

Outline

Very short prehistory of non-euclidean geometry

- The Euclidean postulates were from beginning the object of research due to a long and complicated formulation of the 5th postulate on parallels. Published appx. 300 B.C.
- Many attempts failed until the ideas of non-euclidean geometry appeared in 1829 by N. Lobachevsky, in 1831 by J. Bolyai. In 1824, C. Gauss wrote about such geometry in his letter to his friend F. Taurinus.
- In 1868, E. Beltrami constructed 2-dimensional non-euclidean geometry and introduced pseudosphere (a sphere with negative curvature). The results on hyperbolic geometry started to occur frequently.
- In 1908, H. Minkowski reformulated the famous A. Einstein's paper from 1905 and introduced space-time.



Quadratic form on a vector space

- We consider $\mathbb{R}^n$ together with a quadratic for $\mathbf{q}\colon \mathbb{R}^n \to \mathbb{R}$ given by $\mathbf{q}(\mathbf{x}) = \mathbf{x}^\top Q\mathbf{x}$ with symmetric matrix Q. The corresponding polar form is given by $P(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top Q\mathbf{y}$ and it is a symmetric, bilinear map with relation to the quadratic form given by $\mathbf{q}(\mathbf{x}) = P(\mathbf{x}, \mathbf{x})$. They are mutually unique.
- A space with a quadratic form is called a pseudo-euclidean space. We consider regular $\mathbf{q}$. The form $\mathbf{q}$ plays the role of a square of the norm. The polar form P plays the role of the scalar product in the Euclidean space.
- The possible cases are classified by pairs (p, m), where $p + m = n$ and in some basis of $\mathbb{R}^n$ we can write $\mathbf{q}(\mathbf{x}) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n} x_i^2$. Often denoted by $\mathbb{R}^n_p$.
- The subset $\mathbf{q}^{-1}(0)$ is called *isotropic cone*.

Examples of pseudo-euclidean spaces

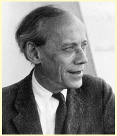
- An example of such space is Euclidean space with *signature* $(n, 0)$.
- An important example is Minkowski space. It is a space $\mathbb{R}^4_1$ with $\mathbf{q}(x_0, x_1, x_2, x_3) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Usually, the $x_0 = ct$-coordinate in physics, where c is the speed of the light (often set to 1 for theoretical reasoning), t is time.
  The isotropic cone consists of so called *light-like* vectors. The vectors outside this cone are either *space-like* vectors, $\mathbf{q}(\mathbf{x}) < 0$ or *time-like* vectors, $\mathbf{q}(\mathbf{x}) > 0$.
- Many properties of the space $\mathbb{R}^{n+1}_1$ are similar to the corresponding properties of the $\mathbb{R}^4_1$ space. For the visualization purposes, we use $\mathbb{R}^3_1$.

Linear automorphism of a pseudo-euclidean space

Felix Klein in his Erlangen Program (1872) proposed a way of classification of geometries (in the sense of the 19th century). For a particular space, we consider invariants of its automorphism group. The idea strongly influences the development of mathematics.

- A linear automorphism f of $\mathbb{R}^n$ such that $\mathbf{q}(\mathbf{x}) = \mathbf{q}(f(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$ is called *quadratic form preserving automorphism*. Similar to the case of transformations of Euclidean space preserving the length of the vectors.
- All such automorphisms form a group denoted $O(p,m) \subseteq GL(n)$ or $O(\mathbf{q})$. The orientation preserving automorphisms are denoted $O^+(p,m)$ or $O^+(\mathbf{q})$. The case $O(1,n)$ will be interesting for us.

Artinian plane



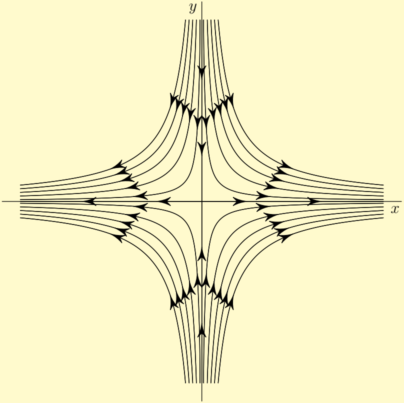
Consider the space $\mathbb{R}^2_1$ with quadratic form $\mathbf{q}(x', y') = x'^2 - y'^2$. Then $\mathbf{q}(x', y') = (x' - y')(x' + y')$. The isotropic cone consists of two lines. Considering the basis $\sqrt{2}/2(1,1)^\top, \sqrt{2}/2(-1,1)^\top$ consisting of the direction of the isotropic lines, we get the form $\mathbf{q}(x, y) = 2xy$. A two-dimensional space with quadratic form of this type in certain basis is called *Artinian*. Then, there are four subsets of matrices representing transformations in the group $O(1,1)$:

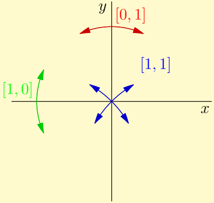
$$\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, k \in \mathbb{R}^*_+,$$

$$\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, k \in \mathbb{R}^*_-, \quad \begin{pmatrix} 0 & k \\ k^{-1} & 0 \end{pmatrix}, k \in \mathbb{R}^*_+, \quad \begin{pmatrix} 0 & k \\ k^{-1} & 0 \end{pmatrix}, k \in \mathbb{R}^*_-.$$

The subset (component) containing identity forms the group of orthochronous transforms $O^{++}(1,1)$.

Orbits of the orthochronous group in the Artinian plane





The orbits of the $(\mathbb{R}^2, 2xy)$ generated by the automorphisms in $O^{++}(1,1)$ – four sets of branches of hyperbolas and four rays. The group $O(1,1)/O^{++}(1,1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein's fourgroup). This structure extends almost directly to higher dimensions.

Angle in the Artinian plane

A map in $O^{++}(1,1)$ can be written in the exponential form

$$\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, k \in \mathbb{R}^*_+, t \in \mathbb{R}. \tag{1}$$

Let $\ell, \ell' \subset \mathbb{R}^2$ be lines. Their angle is defined as the number $\omega \in \mathbb{R}$ such that $\ell' = \Omega(\ell)$, where Ω is given by the matrix $\begin{pmatrix} e^\omega & 0 \\ 0 & e^{-\omega} \end{pmatrix}$. The transformation of the coordinates given by such a matrix is called *hyperbolic rotation*.

The angle ω can also be computed using the cross-ratio $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ on the real projective line generated by $\mathbb{R}^2$. In an affine part of the line, where the points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ have the corresponding coordinate a, b, c, d, we get

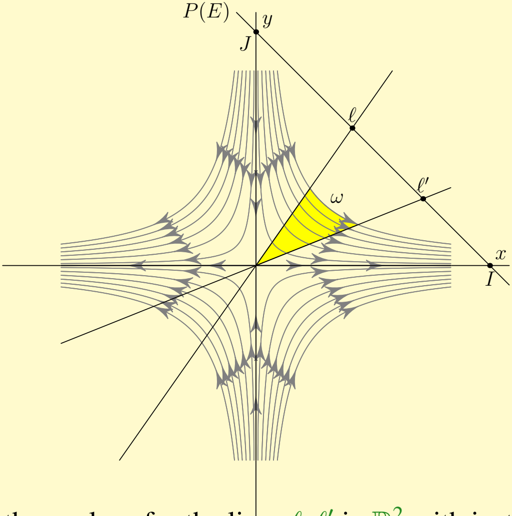
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \frac{a-c}{b-c} : \frac{a-d}{b-d}. \tag{2}$$

Then

$$\omega = \frac{1}{2} \log(\ell, \ell', J, I), \tag{3}$$

where J, I are isotropic lines of $\mathbb{R}^1_1$. It is also ratio of eigenvalues of Ω.

Hyperbolic rotation



The definition of the angle ω for the lines ℓ, ℓ' in $\mathbb{R}^2_1$ with isotropic lines I, J.
The oriented area of the yellow domain is proportional to the angle ω.

Möbius group

- An inversion i of $\mathbb{R}^d \cup \{\infty\}$(homeomorphic to $\mathbb{S}^d$) with respect to $(d-1)$-dimensional hypersphere $S(\mathbf{s}, r)$ is given by formula $i(\mathbf{x}) = \mathbf{s} + r\frac{\mathbf{x}-\mathbf{s}}{\|\mathbf{x}-\mathbf{s}\|^2}, i(\mathbf{s}) = \infty, i(\infty) = \mathbf{s}$. We also add hyperplane reflections of $\mathbb{R}^d$ with $i(\infty) = \infty$. Equivalently, every subsphere of $\mathbb{S}^d$ is mapped to a subsphere of $\mathbb{S}^d$.
- Möbius group $M\ddot{o}b(d)$ is generated by the inversions of $\mathbb{S}^d$ or, equivalently, by the inversions of $\mathbb{R}^{d+1}$ fixing $\mathbb{S}^d$.

The case $d = 2$ can also be described by: $PSL(2, \mathbb{R})$ represented by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ corresponding to the complex function $f(z) = \frac{az+b}{cz+d}$ (homography). Additionally, antihomographies have to be added i.e. $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, a homography composed with reflection using axis of real numbers in $\mathbb{C}$.

Poincaré group of the Minkowski space

- The basic building blocks of the group $O(\mathbf{q})$ for the Minkowski space $\mathbb{R}^{n+1}_1$ are $O(n)$ for the Euclidean space $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}_1$ and $O(1,1)$ for Artinian planes contained in $\mathbb{R}^{n+1}_1$. Moreover for $n \geq 2$
$$O(\mathbf{q}) \simeq M\ddot{o}b(n) \simeq \mathrm{Conf}(\mathbb{D}^n).$$
- The group $O(\mathbf{q})$ can be (upto symmetry in the last n variables) generated by block-diagonal matrices of the type
$$\mathrm{diag}(1, R, I_{n-2}), \mathrm{diag}(H, I_{n-1}), \mathrm{diag}(1, -1, I_{n-1}), \mathrm{diag}(-1, 1, I_{n-1}),$$
where I_k is a identity matrix of type $k \times k$, $R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, and $H = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}$.
- The Poincaré group is a semidirect product of the Lorentz group and the translation group of $\mathbb{R}^{n+1}_1$.
- There is a strong structure similarity of the group $O(\mathbf{q})$ with $O(n)$. Complexification of $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}_1$ produces isomorphic spaces over $\mathbb{C}$.

Transitivity of the action, reflections as generators

Theorem (Witt, 1936)

Let $F, F' \subseteq E$ *be two subspaces and* $f : F \to F'$ *be an isometry with respect to the quadric* $\mathbf{q}$. *Then, there is a map* $\hat{f} \in O(\mathbf{q})$ *such that* $\hat{f}|_F = f$.

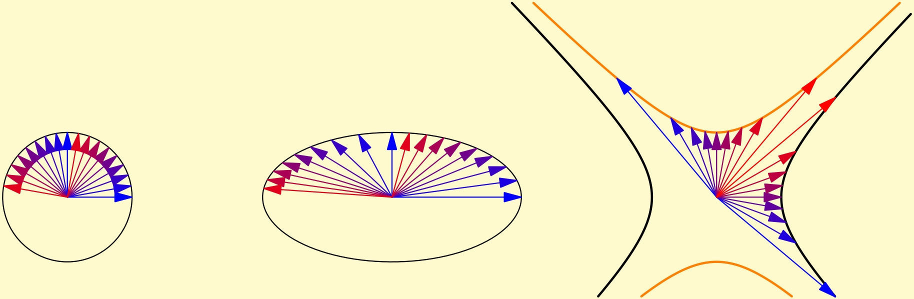
Theorem (Witt – stronger version)

Let $F, F' \subseteq E$ *be two subspaces and* $f : F \to F'$ *be an isometry with respect to the quadric* $\mathbf{q}$. *There is a map* $\hat{f} \in O^+(\mathbf{q})$ *such that* $\hat{f}|_F = f$ *if* $\dim F + dim(\operatorname{rad} F) < \dim E$. *Moreover, if* $\dim F + dim(\operatorname{rad} F) = \dim E$ *and* $f \in O^-(\mathbf{q})$ *then there is no* $g \in O^+(\mathbf{q})$ *such that* $g|_F = f|_F$

Theorem (Cartan–Dieudonné)

Every isometry $f \in O(\mathbf{q})$ *is a product of at most* $\dim E$ *hyperplane reflections.*

Pseudo-orthogonality



Orthogonality of vectors in Euclidean, elliptic and hyperbolic plane. A reflection using linear variety $H \subset \mathbb{R}^n_k$ is given via map

$$f(\mathbf{x}) = \mathbf{x}_H - \mathbf{x}_H^{\perp}, \tag{4}$$

where $\mathbf{x} = \mathbf{x}_H + \mathbf{x}_H^{\perp}$ is a unique decomposition (if possible) of $\mathbf{x}$ such that $\mathbf{x} \in H$ and $\mathbf{x} \in H^{\perp}$.

Special theory of relativity (STR)

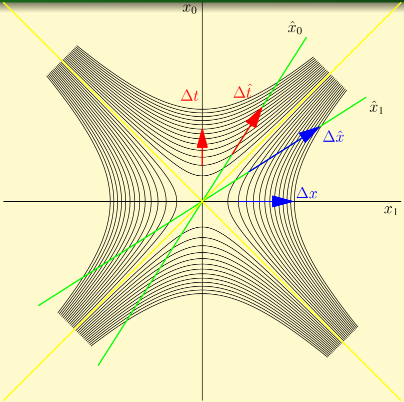
An event is marked with time and space coordinates. Having two observers $O, \hat{O}$ of the same event, we get coordinates (t, x) and $(\hat{t}, \hat{x})$ of the same event. Let the observer $\hat{O}$ is moving by a constant velocity $v < c$ in the direction of the coordinate axis x. Assumptions of STR give Lorentz transformation

$$\begin{aligned} \hat{t} &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}, \\ \hat{x} &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \end{aligned}$$

or alternatively using $\tanh\phi = \frac{v}{c}$ as

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} c\hat{t} \\ \hat{x} \end{pmatrix}. \tag{5}$$

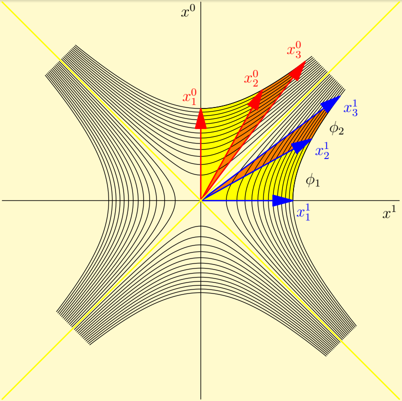
Length contraction, time dilatation



Contraction of length. Let $\hat{A}\hat{B}$ be a segment of unit length in the coordinate system $\hat{O}$. Hence the difference vector has coordinates $(0, 1)$. For the observer O, it has the coordinates $(\sinh\phi, \cosh\phi)$. Hence, the length will be for him shorter by a factor $\cosh\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Dilatation of time. Let $\hat{A}\hat{B}$ be a segment of unit time in the coordinate system $\hat{O}$. Hence the difference vector has coordinates $(1, 0)$. For the observer O, it has the coordinates $(\cosh\phi, \sinh\phi)$. Hence, the time between the events will be for him shorter by a factor $\cosh\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Addition of velocities in STR



Addition of velocities. Let ϕ_1 be the size of the hyperbolic angle between the lines representing the observers O_1, O_2, where O_2 is moving with velocity v_1 with respect to O_1, let ϕ_2 be the size of the hyperbolic angle between the lines representing the observers O_2, O_3,where O_3 is moving with velocity v_2 with respect to O_2 and ϕ_3 be the size of the hyperbolic angle between the observers O_1, O_3, where O_3 is moving with velocity v_3 with respect to O_1. Assuming the directions to be same, we get

$$\phi_3 = \phi_1 + \phi_2 \quad \text{which using} \quad \tanh \phi_i = \frac{v_i}{c} \quad \text{gives} \quad \frac{v_3}{c} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1}{c}\frac{v_2}{c}}. \tag{6}$$

Models of the hyperbolic spaces

- Hyperbolic space is a flat (not curved) space where the hyperplanes (and also all “linear varieties” of lower dimensions) satisfy a special set of rules for parallelity, different from the usual axioms of Euclidean geometry.
- The models of the hyperbolic spaces of dimension n can be obtained using $\mathbb{R}^{n+1}_n$ with $\mathbf{q}(\mathbf{x}) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. The hypersurface $\mathbf{q}^{-1}(-1)$ (the “hypersphere of radius -1”) can serve as the example.
- We recall well known models for hyperbolic geometry – the hyperboloid model, the Beltrami-Klein’s model and the Poincaré’s models. We consider the case of plane, since it can be drawn, however the same construction works for arbitrary finite dimension. The distance in the models can be defined using projective geometry.

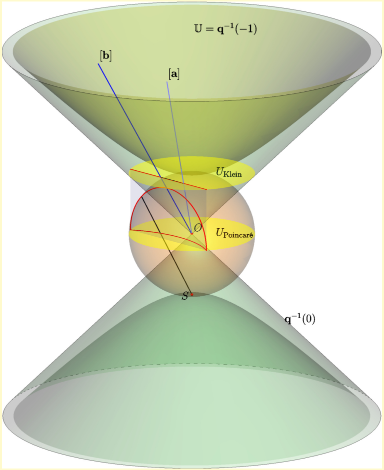
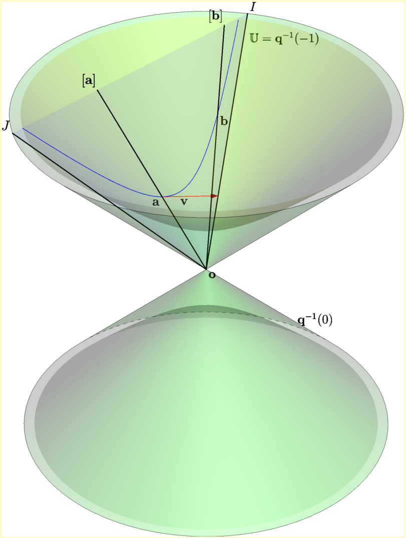


Figure: The locations of the models of hyperbolic geometry in Minkowski space.

The hyperboloid model

The points of this model are on the upper part of the hyperboloid $\mathbb{U} = \mathbf{q}^{-1}(-1) \cap \{\mathbf{x} \in \mathbb{R}_n^{n+1}, x_{n+1} > 0\}$. The tangent space of each point inherits its positive definite scalar product from $\mathbb{R}_n^{n+1}$. The distance between points $\mathbf{a}, \mathbf{b}$ is computed as $\int_{t_\mathbf{a}}^{t_\mathbf{b}} \sqrt{\mathbf{q}(\dot{\mathbf{x}}(t))}\mathrm{d}t$ with $\mathbf{x}(t)$ given by the intersection of $\mathbb{U} \cap [\mathbf{a}, \mathbf{b}]$. It can be parameterized by $\mathbf{a}\cosh(t) + \mathbf{v}\sinh(t), t \in \mathbb{R}$, $\mathbf{q}(\mathbf{v}) = 1$ (analogous to the Euclidean case). This is also geodesic line on $\mathbb{U}$ between those two points. The space $\mathbb{U}$ is complete, constantly curved and its isometries are formed by the orthochronous subgroup of $M\ddot{o}b(n)$.



Weierstrass model

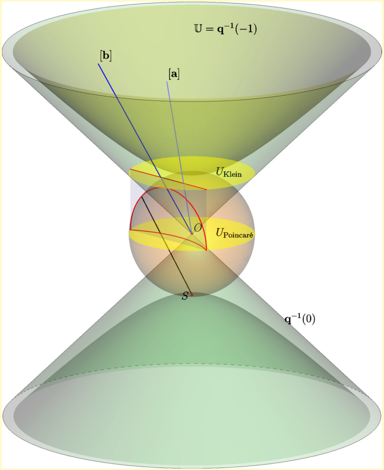
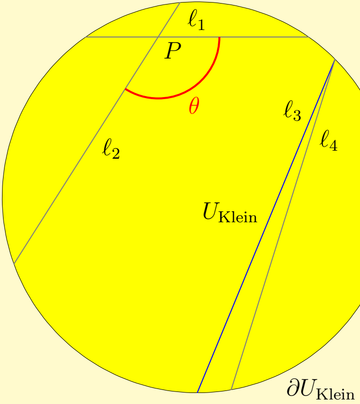


Figure: The locations of the models of hyperbolic geometry in Minkowski space.

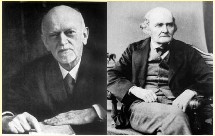
Beltrami-Klein's model

The points of the n-dimensional hyperbolic space are given as inner points of the unit radius disk in $\mathbb{D}^n(\mathbf{0}, 1) \subset \mathbb{R}^n$ corresponding to one-dimensional subspaces of $\mathbb{R}^{n+1}$ with negative $\mathbf{q}$. The linear subvarieties of the hyperbolic space are intersections of the affine varieties in $\mathbb{R}^n$ with $\operatorname{int} \mathbb{D}^n$. The distance in the model is mapped from the hyperboloid model. The Euclidean angles between affine spaces do not represent the hyperbolic angles. The model is also called projective.



Beltrami-Klein's model of the hyperbolic plane.

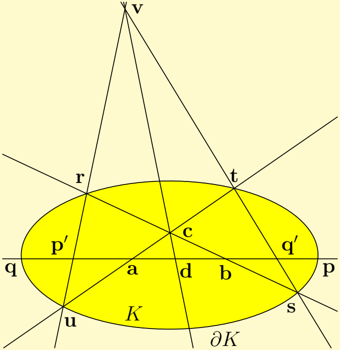
Hilbert's metric



Hilbert (Cayley-Klein) metric h in the bounded convex domain K is given by the hyperbolic angle via

$$h(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \log(\mathbf{a}, \mathbf{b}, \mathbf{p}, \mathbf{q}).$$

The projective invariance and the multiplicative properties of the cross-ratio give the triangle inequality $h(\mathbf{a}, \mathbf{c}) + h(\mathbf{c}, \mathbf{b}) \geq h(\mathbf{a}, \mathbf{b})$.



- $(\mathbf{a}, \mathbf{d}, \mathbf{p}, \mathbf{q})(\mathbf{d}, \mathbf{b}, \mathbf{p}, \mathbf{q}) = (\mathbf{a}, \mathbf{b}, \mathbf{p}, \mathbf{q})$
- $(\mathbf{a}, \mathbf{c}, \mathbf{t}, \mathbf{u}) = (\mathbf{a}, \mathbf{d}, \mathbf{p}', \mathbf{q}') \geq (\mathbf{a}, \mathbf{d}, \mathbf{p}, \mathbf{q})$
- $(\mathbf{c}, \mathbf{b}, \mathbf{s}, \mathbf{r}) = (\mathbf{d}, \mathbf{b}, \mathbf{p}', \mathbf{q}') \geq (\mathbf{d}, \mathbf{b}, \mathbf{p}, \mathbf{q})$

The identification of such metric was also considered in the more general context of the 4th Hilbert problem (identification of metric with straight line geodesics).
For non-bounded convex domain, a slight modification e.g. with Euclidean distance is used.

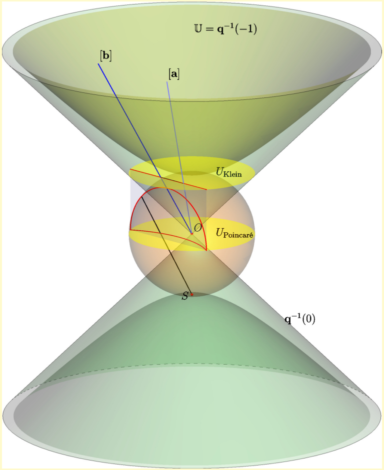


Figure: The locations of the models of hyperbolic geometry in Minkowski space.

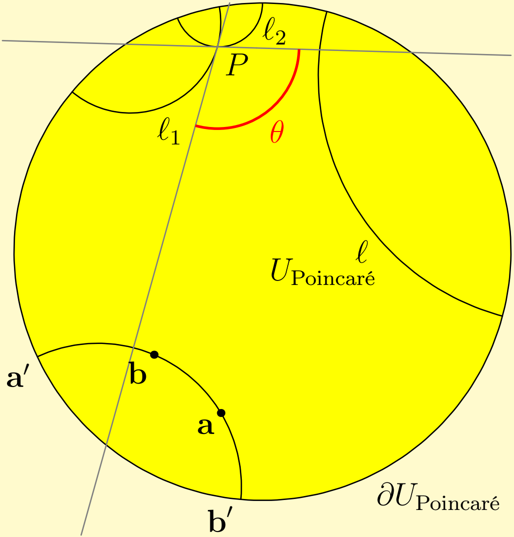
Poincare's model

The Beltrami-Klein model is not conformal. There is a way to improve the model using a modified mapping from the hyperboloid model $\mathbb{U}$.
Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere and $\mathbb{S}^n_+$ be its upper half-sphere. It encloses the disk $\mathbb{D}^n(\mathbf{0}, 1)$ via its "equator" $\mathbb{S}^{n-1} = \mathbb{S}^n \cap \mathbb{R}^n$. The map $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$ restricted to the $\mathbb{S}^n_+$ is bijective. Let $\rho\colon \mathbb{S}^n - \{S\} \to \mathbb{R}^n$ be the stereographic projection from the "south pole" $S = (0, \dots, 0, -1)$. Then the conformal model can be get using maps $\rho \circ (\pi|_\Theta)^{-1}\colon \mathbb{D}^n \to \mathbb{D}^n$.
The group of isometries agrees with

$$\mathrm{Conf}(\mathbb{D}^n) = \{Ac(\mathbf{x}) : A \in O(n), c \in \{Id_{\mathbb{D}^n}, inv_{\perp\partial\mathbb{D}^n}\}\}.$$

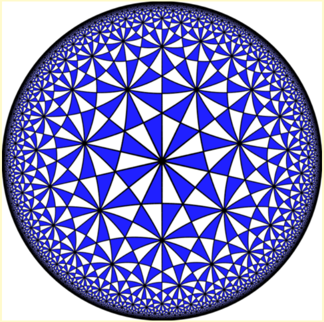
Using a special inversion of $\mathbb{R}^n$, one gets also Poincaré halfspace model.

Poincaré disc model



Area of a polygon in hyperbolic plane

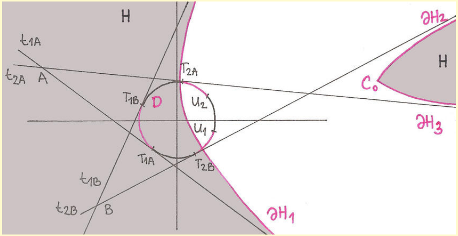
- Using Gauss-Bonet theorem, the area of the triangle in hyperbolic space is $\pi - \alpha - \beta - \gamma$ (upto a constant multiple, curvature). The area of a polygon is hence $(n-2)\pi - \sum_{i=1}^{m} \alpha_i$
- there are infinitely many non-isomorphic discrete subgroups of $O(\mathbf{q})$ generating tiling of the hyperbolic space starting with a properly chosen polygon (some were used in the M. C. Escher paintings)

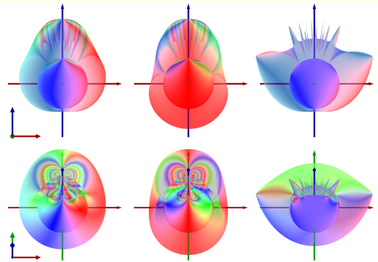


More topics and applications

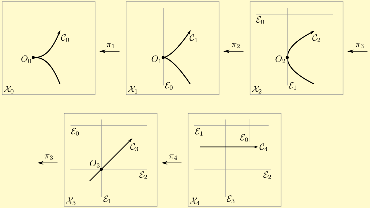
- classical hyperbolic geometry – synthetic/analytic properties
- modern topology/geometry of hyperbolic spaces/manifolds
- theory of relativity
- discrete subgroups of $M\ddot{o}b(n)$
- geometric modeling/approximations/(applied) algebraic geometry
  - medial axis transform – way of representing compact sets T having property $\mathrm{cl}(\mathrm{int}\, T) = T$
  - Minkowski Phytagorean hodograph curves
  - many computational structures (e.g. Voronoi diagrams)
  - rendering in hyperbolic space

Ongoing work connected to this area





- classification of polynomial/rational curves with respect to the **q** based on control points (Barbora Pokorná)
- visualizations of complex functions (Miroslava Valíková)
- structure of isolated singularities and their deformations (Martina Bátorová)



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Habilitation lecture

THANKS FOR YOUR ATTENTION