International Journal of Modelling and Simulation, Vol. 24, No. 2, 2004

MODELLING WITH THREE TYPES OF COONS BODIES

R. Ďurikovič* and S. Czanner*

Abstract

This article discusses the mathematical concepts of multivariate parametric solids and their description by B-spline basis functions. Parametric solids can model both the shape and unisotropic interior. Three types of parametric solids, Coons body of types 0, 1, and 2, are used to demonstrate the interior modelling often used as initial or boundary conditions in numerical simulation. Multiple parametric solid elements can be joined together to form a complicated shape. Continuity between elements can be defined as in B-spline modelling. The proposed methodology and modelling technique is applied to a metamorphosis of two given 3-D shapes.

Key Words

Coons body, multivariate parametric solids, B-spline basis functions

1. Introduction

Many unambiguous solid representation techniques, such as primitive instancing, cell decomposition, constructive solid geometry, sweep and boundary or medial axis representation [1] have the limitation that they do not offer ways to represent internal behaviour. The representation is considered to be unambiguous when it corresponds to one and only one object in the object space. Unfortunately, the more recent and modern technique, the real function representation [2], does not solve this problem either. The above techniques assume an internal homogenity of the model. Nevertheless, they are adequate for many simulations and design applications. Still more and more complicated physical models, for which scalar, vector, and tensor-valued physical fields are needed, increase the demand on modelling of both the shape and the distribution of fields as initial or boundary conditions for simulation. Some of the applications include areas of structural mechanics and ablation thermo- and aerodynamics [3] and the description of inhomogeneous materials. The representation that offers interior modelling is a parametric function representation: $F\colon R^n \to R^m$, where R^n and R^m are the parameter and object spaces, respectively. In the trivariate case, the natural extension to triangles and rectangles are tetrahedra, pentahedra, and hexahedra, which are extensively used for Lagrange and Hermite basis functions in the finite element (FEM) literature [4].

Sederberg and Parry [5] introduced the so-called Free-Form Deformation (FFD). This method imposes an initial deformation lattice on a parallelepiped, and defines the deformable space as the trivariate Bezier volume defined by lattice points. Griessmain and Purtgathofer [6] modified the technique by utilizing a B-spline volume representation.

This article, in Section 2, introduces the mathematical concepts of multivariate *parametric solids*, and extents the definition of a Coons patch to Coons solids. Three types of parametric solids, Coons body 0, 1, and 2, are used to demonstrate the interior modelling often used as initial or boundary conditions in numerical simulation. In Section 3 we modify the control points of boundary surfaces and curves to model the interior and the shape of objects. Modelling with multiple Coons solids is also discussed here. Finally, an application of the proposed technique and the object metamorphosis is discussed in Section 4.

2. Three Construction Steps of Coons Body

We will restrict ourselves to the parametric function representation: $F\colon R^3 \to R^3$. The well-known Coons patch is generalized to tensor-product Coon parametric solids called here body 0, 1, and 2. Each of these three representations has a different ability to control the interior of a parametric solid. The simplest and most limited is *Coons body* 0 where only the shape of a quadrilateral can be changed. *Coons body* 1 allows us to control the interior by modifying the control points along the edge curves of a parametric solid. The most general is *Coons body* 2, having the ability to modify the shape and interior, using points within the boundary surfaces of a parametric solid.

In general, a *tensor-product solid* of degree (l, m, n) is defined to be:

$$\mathbf{p}(u,v,w) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{b}_{ijk} F_i^l(u) F_j^m(v) F_k^n(w)$$

where $u, v, w \in [0,1]$, $\mathbf{b}_{ijk} \in R^3$, and F_i are basis functions. Notice that for a volume of degree $(3,3,3)$, there are $4 \times 4 \times 4 = 64$ algebraic vectors and 64 geometric vectors, so there are 192 coefficients (see Fig. 1).

* Department of Computer Software, University of Aizu, Aizu-Wakamatsu, Fukushima-ken 965-8580, Japan; e-mail: {roman, czanner}@u-aizu.ac.jp

Recommended by Dr. C. Putcha
(paper no. 205-3029)

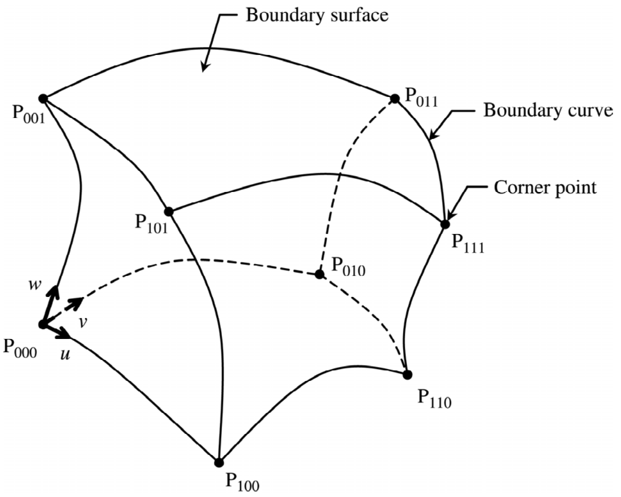


Figure 1. Tensor-product solid of degree (3, 3, 3).

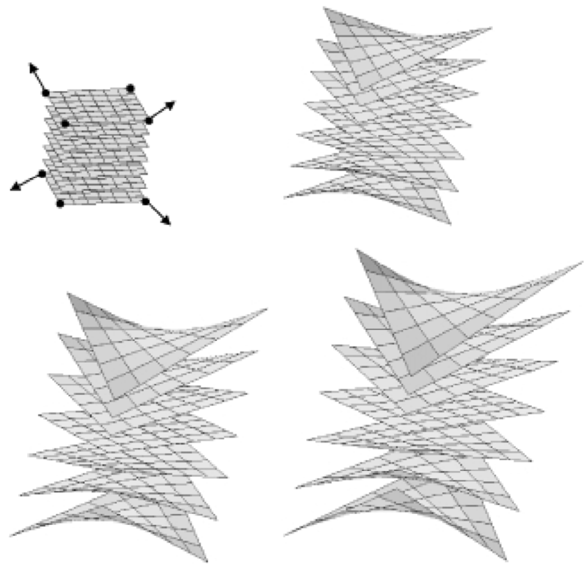


Figure 2. Coons body 0: shape morphing of original cube by eight control points.

2.1 Visualization

Simple mappings $F\colon R^3 \rightarrow R^3$ with small polynomial degrees can be understood by displaying a set of isoparametric surfaces of the object (see Fig. 2). One way to get a good idea of the shape of an object is to display several different images. For example, we can visualize the shaded shape with projected parametric curves and display three parametric surfaces corresponding to constant u, v, and w. To show the shape and interior changes in time, that is, mappings $F\colon R^4 \rightarrow R^4$, time animation of isoparametric surfaces and outer shape is one possibility. The visualization techniques just mentioned are also used in this work.

2.2 Coons Body 0

Coons body 0 is a trilinear interpolation of eight points P_{ijk}, i, j, $k \in \{0, 1\}$, defined as:

$$T(P_{ijk}; u, v, w) = (1-u, u) \times \begin{pmatrix} (1-v, v) \begin{pmatrix} P_{000} & P_{001} \\ P_{010} & P_{011} \end{pmatrix} \begin{pmatrix} 1-w \\ w \end{pmatrix} \\ (1-v, v) \begin{pmatrix} P_{100} & P_{101} \\ P_{110} & P_{111} \end{pmatrix} \begin{pmatrix} 1-w \\ w \end{pmatrix} \end{pmatrix}$$

where u, v, $w \in [0, 1]$.

Coons body 0 has some obvious properties that can be derived from the tensor-product solid:

- *Derivatives*: Because of the linear independence of u, v, w coordinates, the partial derivatives with respect to u, v, w have obvious geometric interpretation, and they coincide with the derivatives along parametric lines.
- *Boundary surfaces*: There are six hyperbolic paraboloids on the boundary, namely, $T(P_{ijk}; 0, v, w)$, $T(P_{ijk}; 1, v, w)$, $T(P_{ijk}; u, 0, w)$, $T(P_{ijk}; u, 1, w)$, $T(P_{ijk}; u, v, 0)$, and $T(P_{ijk}; u, v, 1)$.
- *Boundary curves*: The boundary curves of Coons body 0 are linear segments. Setting two parameters of u, v, and w equal to 0, or 1, we get 12 linear segments, $T(P_{ijk}; 0, 0, w)$, $T(P_{ijk}; 0, 1, w)$, ...
- *Control grid*: The control grid points P_{ijk} coincide with eight vertices of the quadrilateral.

An example of Coons body 0 element is shown in Fig. 2. A simple cube is deformed by moving four corner points in upward and downward directions. As a result, the shape of the body changes but the interior "density" cannot be modified. This element is often used in FEM methods when only linear basis functions are used within a 3D element.

2.3 Coons Body 1

Let us assume the compatible curves $C(u, j, k)$, $C(i, v, k)$, and $C(i, j, w)$; $i, j, k \in \{0, 1\}$; $u, v, w \in [0, 1]$, with the following property:

$$\begin{aligned} C(u, j, k)_{u=i} &= C(i, v, k)_{v=j} = C(i, j, w)_{w=k} \\ &= C(i, j, k) =: C_{ijk} \end{aligned}$$

for each $i, j, k \in \{0, 1\}$.

Coons body 1 is a bilinear interpolation between pairs of compatible curves defined as:

$$\begin{aligned} &C1(u, v, w) \\ &= (1-u, u) \begin{pmatrix} C(0,0,w) & C(0,1,w) \\ C(1,0,w) & C(1,1,w) \end{pmatrix} \begin{pmatrix} 1-v \\ v \end{pmatrix} \\ &\quad + (1-u, u) \begin{pmatrix} C(0,v,0) & C(0,v,1) \\ C(1,v,0) & C(1,v,1) \end{pmatrix} \begin{pmatrix} 1-w \\ w \end{pmatrix} \\ &\quad + (1-v, v) \begin{pmatrix} C(u,0,0) & C(u,0,1) \\ C(u,1,0) & C(u,1,1) \end{pmatrix} \begin{pmatrix} 1-w \\ w \end{pmatrix} \\ &\quad - 2T(C_{ijk}; u, v, w) \end{aligned}$$

Coons body 1 has the following properties:

- *Derivatives*: The partial derivatives with respect to u, v, w coincide with the derivatives along parametric lines.
- *Boundary surfaces*: There are six Coons patches on the boundary derived from four-tuples of boundary curves.
- *Boundary curves*: The boundary curves of Coons body 1 are compatible curves $C(u, j, k)$, $C(i, v, k)$, and $C(i, j, w)$ [7]. In the simplest case, when the boundary curves are linear, the Coons body 1 is actually the Coons body 0.
- *Vertices*: Vertices of $C1(u, v, w)$ are the points C_{ijk}.
- *Control grid*: Control grid points coincide with the control points of the boundary curves.

An example of Coons body 1 shown in Fig. 3 demonstrates the cubic element defined by B-spline boundary curves each having five control points. The B-spline de Boor points are distributed along the edges of a simple cube, and only the points with assigned arrows in the image are used to deform the shape. The boundary curves of this solid can define the influence of boundary surfaces on the interior without changing the outer shape. The effect of density modification is shown in Fig. 4; the left image shows the original body with a few parametric surfaces and the middle image shows the solid with a modified interior. It can be observed that the interior in the right image deforms much earlier than that in the left image. The right image of Fig. 4 is the outer shape that remains constant during the interior modelling. The interior modelling is still limited in Coons body 1 as we cannot directly modify the centre of boundary surfaces.

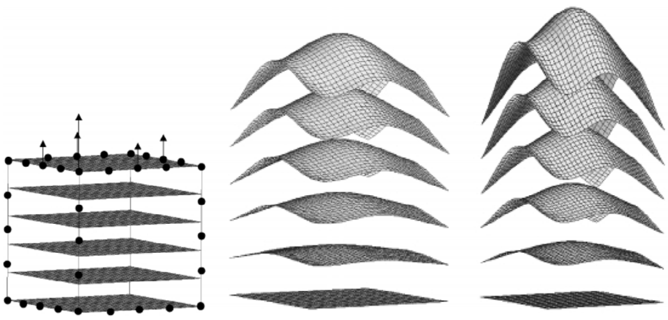


Figure 3. Coons body 1 with B-spline boundary curves. B-spline de Boor points along the edges deform the cube.

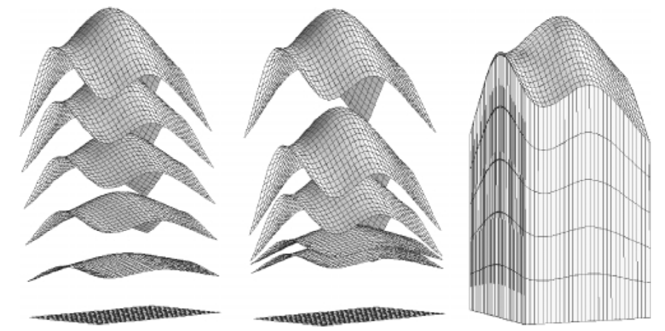


Figure 4. Coons body 1: preserving the shape while changing the interior density.

2.4 Coons Body 2

Let us assume three couples of arbitrary opposite patches $Z_k(u,v) = C(u,v,k)$; $Y_j(u,w) = C(u,j,w)$; $X_i(v,w) = C(i,v,w)$; $i,j,k \in \{0,1\}$, $u,v,w \in [0,1]$, with common boundary curves as stated below:

$$
\begin{aligned}
C(u,v,k)_{v=j} &= C(u,j,w)_{w=k} =: C(u,j,k); j,k = 0,1\\
C(u,v,k)_{u=i} &= C(i,v,w)_{w=k} =: C(i,v,k); i,k = 0,1\\
C(i,v,w)_{v=j} &= C(u,j,w)_{u=i} =: C(i,j,w); i,j = 0,1
\end{aligned}
$$

To make things simpler, the connected patches should have no other intersections, for example, $C(u,j,k) = Y_j(u,w) \cap Z_k(u,v)$, $C(i,v,k) = X_i(v,w) \cap Z_k(u,v)$ and $C(i,j,w) = X_i(v,w) \cap Y_j(u,w)$, $i,j,k \in \{0,1\}$.

Coons body 2 is defined to be:

$$C2(u,v,w) = D(u,v,w) - C1(u,v,w) - T(C_{ijk}; u,v,w)$$

where $T(C_{ijk}; u,v,w)$ and $C1(u,v,w)$ are Coons bodies 0 and 1, respectively; $D(u,v,w)$ is a linear interpolation between pairs of opposite patches given by:

$$
\begin{aligned}
D(u,v,w) = &(1-u,u)\begin{pmatrix} C(0,v,w) \\ C(1,v,w) \end{pmatrix}\\
&+(1-v,v)\begin{pmatrix} C(u,0,w) \\ C(u,1,w) \end{pmatrix}\\
&+(1-w,w)\begin{pmatrix} C(u,v,0) \\ C(u,v,1) \end{pmatrix}
\end{aligned}
$$

Coons body 2 has the following properties:

- *Derivatives*: The partial derivatives with respect to u, v, w coincide with the derivatives along parametric lines.
- *Boundary surfaces*: There are six boundary faces $X_i(v,w)$, $Y_j(u,w)$, and $Z_k(u,v)$, used in the definition [7]. Note that if the patches are the Coons patches defined by four-tuples of boundary curves, then the Coons body 2 is actually the Coons body 1. In the simplest case, where patches have linear boundary curves, the Coons body 0 is obtained.
- *Boundary curves*: The boundary curves of Coons body 2 are common boundary curves $C(u,j,k)$, $C(i,v,k)$, and $C(i,j,w)$.
- *Vertices*: Vertices of $C2(u,v,w)$ are the points C_{ijk}.
- *Control grid*: Control grid points coincide with the control grid points of boundary surfaces.

An example of Coons body 2 using B-spline boundary surfaces is shown in Fig. 5. The de Boor control points of this body are distributed at corners, on edges, and within the interior of boundary surfaces. De Boor control vertices distributed over the B-spline surfaces create the shape folding and deform a simple cube in Fig. 5. Additional freedom in modelling is gained from control points distributed within the centre of B-spline patches. It can be shown that the Coons body 2 defined by the B-spline boundary surfaces is a B-spline body [7]. A nail represented as a single Coons body 2, illustrated in Fig. 6, has two different interior densities while preserving the same outer shape. The right image shows the shaded shape and surface parametric curves of a nail. Moving control points belonging to edges along the nail body modify its interior density.

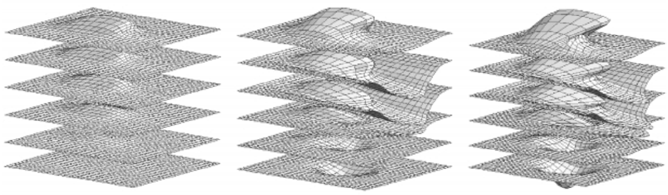


Figure 5. Coons body 2 with B-spline boundary surfaces. The de Boor points of boundary surfaces can deform the shape and interior of the original cube.

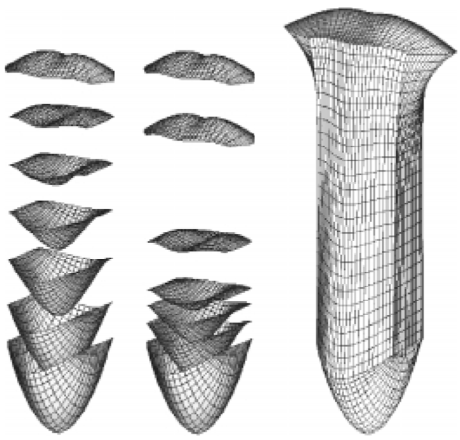


Figure 6. Nail as Coons body 2 with two different interior density distributions.

3. Modelling with Coons Bodies

A lattice is defined as a set of points that generates a volume. A control lattice has vertices, edges, faces, and cells. In the case of Coons bodies, each cell of the control lattice is defined by six faces and each face by four vertices. Each vertex has connectivity six. Each cell defines a Coons body (volume). Because any Coons body 2 is a B-spline volume, the shape continuity is simply controlled in a similar way as is done for B-splines. Similarly, multiple control points can also be used in Coons body 2 to create sharp edges and corners.

In an example shown in Fig. 7, the modelling of a spool starts from a cube. First, the bottom of a cube is rounded with control points on the bottom side of a cube; next the top is created by modifying the control points on top side. Top and bottom points also define the thickness of the spool. Other control points are not modified. The control grid and the shaded surfaces for two steps of spool modelling are shown in Fig. 7.

Multiple Coons bodies 2 combined with set-theoretic operations can create complicated shapes. In Fig. 8, three bodies with the union set operation are used to form the final shape. In the case when blending between several parts is required, a single Coons body 2 with an extended number of control points can be used. Five Coons bodies 2 were used to design an ergonomic chair in Fig. 9. The control grid used for modelling is shown in the right image.

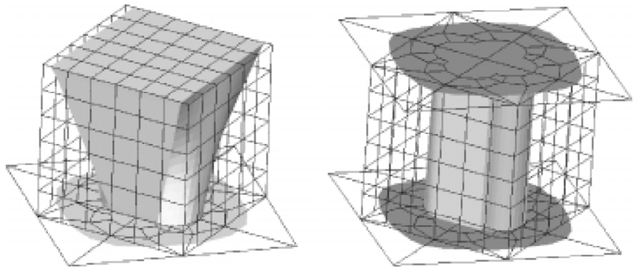


Figure 7. Modelling a spool as a Coons body 2 starting from a cube.

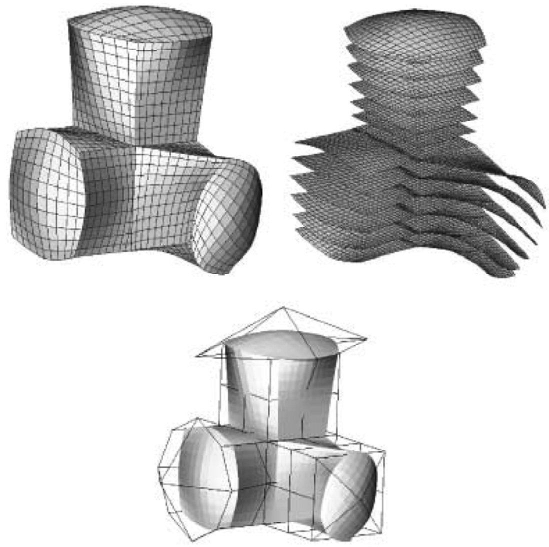


Figure 8. Union set operation between three Coons bodies 2.

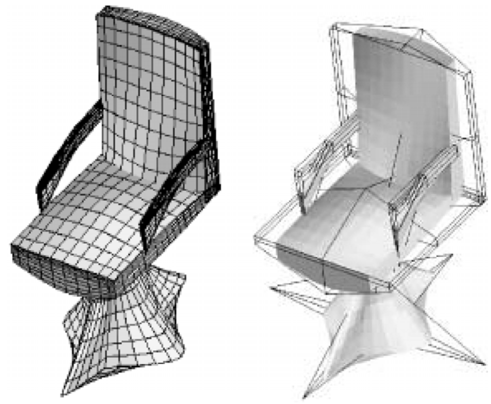


Figure 9. Chair represented with five Coons bodies 2 and its control grid.

4. Metamorphosis with Coons Bodies

In the case of Coons bodies, we are dealing, for example, with the deformation of an unit cube in the sense that we operate on the material forming the cube to alter its shape by pressing, drawing, twisting, or turning. This deformation process has local character within the body. In general, the position of every point in the interior as well as those on the surface is altered when a grid point is moved. In the case where two different sets of control points define the same boundary surface (except the parametrization), the shape of the body does not change but the interior will change (see Figs. 4 and 6). In order to carry out the metamorphosis of two objects represented by a Coons body, for every grid point of the first object the associated grid point from the other object must be found. Then the grid deformation, by moving a grid point along the path, defines the metamorphosis, similar to the motion path in animation. Metamorphosis between the cube and the spool, shown in Fig. 10, uses the grid deformation technique.

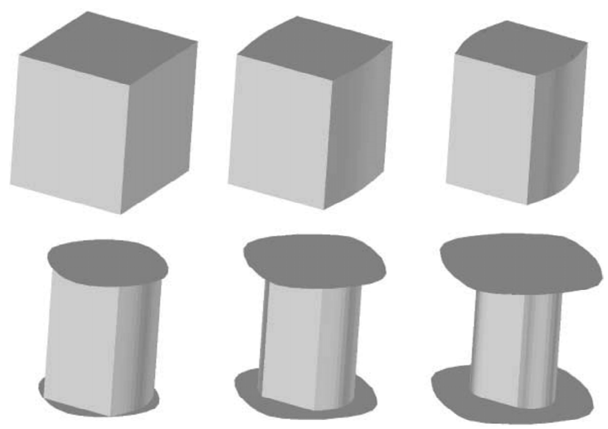


Figure 10. Metamorphosis from cube to spool using the Coons body 2.

5. Conclusion

The proposed system implemented under the Linux Red Hat 6.2 operating system uses the parametric Coons bodies to model the interior and outer shape of a body. Presented examples show how easy it is to define and change the interior. Natural extensions to interior modelling are the shape deformation and metamorphosis discussed in this work.

The authors see the advantage of this technique as being the ability to define the boundary and internal initial conditions prior to numerical simulations. The method works well for symmetric shapes used in engineering simulation. Parametric solids can be effectively applied in finite elements with irregular or *trimmed boundaries* to avoid difficult problems in model design for FEM.

The proposed method was tested in shape deformation examples using both engineering and organic shapes. It was quite difficult to apply the parametric solids for representing and deforming organic shapes. Coons bodies can be easily utilized in FFD applications. Better results of shape deformation could be obtained using the fifth-dimensional parametric solids under current investigation.

Acknowledgment

The authors thank V. Zatko, Z. Mederlyová, and M. Mikita for their active discussions during the course of this work. This research was sponsored by grants from the Fukushima Prefectural Foundation in Japan for the Advancement of Science and Education.

References

[1] A.A.G. Requicha, Representation of rigid solids: theory, methods, and systems, *Computing Surveys*, *12*, 1980, 437–465.

[2] A. Pasko, V. Adzhiev, A. Sourin, & V. Savcenko, Function representation in geometric modeling: Concepts, implementation and applications, *The Visual Computer*, *11*, 1995, 429–446.

[3] E.L. Stanton, L.M. Crain, & T.F. New, A parametric cubic modeling system for general solids of composite material, *International Journal of Numerical Methods in Engineering*, *11*, 1977, 653–670.

[4] J.C. Sabonnadiere & J.L. Coulomb, Finite element methods in CAD: Electrical and magnetic fields (Berlin: Springer, 1987).

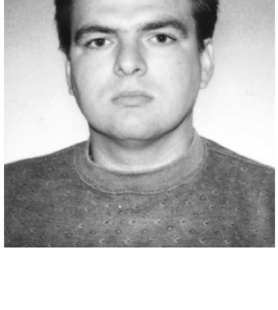
[5] T.W. Sederberg & S.R. Parry, Free-form deformation of solid geometric models, *Computer Graphics (SIGGRAPH '86)*, *20*, Dallas, Texas, 1986, 151–160.

[6] J. Griessmair & W. Purtgathofer, Deformation of solids with trivariate B-splines, *Proc. Eurographics '98 Conf.*, Lisbon, Portugal, September 1989, 137–148.

[7] V. Zatko & Z. Mederlyová, On the coons bodies, *Proc. 15th Spring Conf. on Computer Graphics (SCCG 1999)*, Budmerice, Slovakia, May 1999, 247–253.

Biographies

Roman Ďurikovič obtained his RNDr. in numerical analysis from Comenius University in Slovakia 1998, Ph.D. in computer science from the Hiroshima University in Japan in 1996. He has been an invited researcher at the Kyushu University, Japan. Currently he is an Assistant Professor of Computer Graphics at the University of Aizu in Japan. His research interests include graphics, rendering with GPU, shape modeling, and physical-based animations. Dr. Ďurikovič serves as reviewer for following conferences: CGI Japan, SCCG Slovakia, WSCG Czech Republic, and EUROGRAPHICS.



Silvester Czanner received his M.Sc. and Ph.D. in mathematics from Comenius University in Slovakia in 1994 and 1998. After receiving his Ph.D., he has been a visiting researcher at the University of Aizu. His research interests include application of computer graphics in medicine, organ growth simulation and visualization.