Bezier curve and surface derivations

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Bezier curve: $Q_n(t) := \sum_{k=0}^{n} P_k Be_k^n(t)$

Bernstein polynom: $Be_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k}$

Derivation of Bezier curve:

$$\begin{aligned} Q_n'(t) &= \textstyle\sum_{k=0}^{n} P_k \Big(Be_k^n(t)\Big)' \\ &= \textstyle\sum_{k=0}^{n} \binom{n}{k} P_k \Big(t^k(1-t)^{n-k}\Big)' \\ &= \textstyle\sum_{k=0}^{n} \binom{n}{k} P_k \Big(kt^{k-1}(1-t)^{n-k} - (n-k)t^k(1-t)^{n-k-1}\Big) \end{aligned}$$

let ...
$S_0^n(t) := P_0 t^0 (1-t)^n = -nP_0(1-t)^{n-1}$
$S_k^n(t) := \binom{n}{k} P_k \Big(kt^{k-1}(1-t)^{n-k} - (n-k)t^k(1-t)^{n-k-1}\Big)$ for $k \in \{1, 2, \cdots, n-1\}$
$S_n^n(t) := P_n t^n (1-t)^0 = nP_n t^{n-1}$

so ...

$$(k+1)\binom{n}{k+1} = (k+1)\frac{n!}{(k+1)!(n-k-1)!} = n\frac{(n-1)!}{k!(n-1-k)!} = n\binom{n-1}{k}$$
$$(n-k)\binom{n}{k} = (n-k)\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{k!(n-1-k)!} = n\binom{n-1}{k}$$

$S_k^n(t) + S_{k+1}^n(t) =$
$= \binom{n}{k} P_k \Big(kt^{k-1}(1-t)^{n-k} - (n-k)t^k(1-t)^{n-k-1}\Big) + \binom{n}{k+1} P_{k+1}\Big((k+1)t^k(1-t)^{n-k-1} - (n-k-1)t^{k+1}(1-t)^{n-k-2}\Big)$
$= k\binom{n}{k} P_k t^{k-1}(1-t)^{n-k} + \Big((k+1)\binom{n}{k+1}P_{k+1} - (n-k)\binom{n}{k}P_k\Big)t^k(1-t)^{n-k-1} - (n-k-1)\binom{n}{k+1}P_{k+1}t^{k+1}(1-t)^{n-k-2}$
$= k\binom{n}{k} P_k t^{k-1}(1-t)^{n-k} + n\binom{n-1}{k}\Big(P_{k+1} - P_k\Big)t^k(1-t)^{n-k-1} - (n-k-1)\binom{n}{k+1}P_{k+1}t^{k+1}(1-t)^{n-k-2}$

hence ...
$S_k^n(t) + S_{k+1}^n(t) + S_{k+2}^n(t) =$
$= k\binom{n}{k} P_k t^{k-1}(1-t)^{n-k} + n\binom{n-1}{k}\Big(P_{k+1} - P_k\Big)t^k(1-t)^{n-k-1} + n\binom{n-1}{k+1}\Big(P_{k+2} - P_{k+1}\Big)t^{k+1}(1-t)^{n-k-2} -$
$(n-k-2)\binom{n}{k+2}P_{k+2}t^{k+2}(1-t)^{n-k-3}$

and therefore ...
$\sum_1^{n-1} S_k^n(t) = \binom{n}{1}P_1 t^0(1-t)^{n-1} + n\sum_1^{n-2}\binom{n-1}{k}\Big(P_{k+1} - P_k\Big)t^k(1-t)^{n-1-k} - \binom{n}{n-1}P_n - 1t^{n-1}(1-t)^0$

and let ...

$$P_k' := P_{k+1} - P_k$$

and finally ...
$Q_n'(t) = \sum_0^n S_k^n(t) = n\sum_0^{n-1}\binom{n-1}{k}\Big(P_{k+1} - P_k\Big)t^k(1-t)^{n-1-k} = n\sum_0^{n-1}\Big(P_{k+1} - P_k\Big)Be_k^{n-1}(t) = n\sum_0^{n-1} P_k' Be_k^{n-1}(t)$

$\mathbf{n^{th}}$ Derivation of Bezier curve:

$$P_k^{(0)} := P_k$$
$$P_k^{(1)} := P_{k+1}^{(0)} - P_k^{(0)} = P_{k+1} - P_k$$
$$P_k^{(2)} := P_{k+1}^{(1)} - P_k^{(1)} = P_{k+2}^{(0)} - P_{k+1}^{(0)} - (P_{k+1}^{(0)} - P_k^{(0)}) = P_{k+2}^{(0)} - 2P_{k+1}^{(0)} - P_k^{(0)}$$
$$\vdots$$
$$P_k^{(m)} := P_{k+1}^{(m-1)} - P_k^{(m-1)} = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} P_{k+i}$$

so ...

$$Q_n^{(0)}(t) := \sum_{k=0}^{n} P_k^{(0)} Be_k^n(t)$$
$$Q_n^{(1)}(t) := n \sum_{k=0}^{n-1} P_k^{(1)} Be_k^{n-1}(t)$$
$$Q_n^{(2)}(t) := n(n-1) \sum_{k=0}^{n-2} P_k^{(2)} Be_k^{n-2}(t)$$
$$\vdots$$
$$Q_n^{(m)}(t) := \frac{n!}{m!} \sum_{k=0}^{n-m} P_k^{(m)} Be_k^{n-m}(t)$$

Bezier surface: $Q_{m,n}(u,v) := \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} Be_i^m(u) Be_j^n(v)$

$$\frac{\partial P_{ij}}{\partial i} := P_{i+1,j} - P_{i,j}$$
$$qu_{m,n}(u,v) = \frac{\partial Q_{m,n}(u,v)}{\partial u} := \sum_{i=0}^{m-1} \sum_{j=0}^{n} \frac{\partial P_{i,j}}{\partial i} Be_i^{m-1}(u) Be_j^n(v)$$
$$\frac{\partial P_{ij}}{\partial j} := P_{i,j+1} - P_{i,j}$$
$$qv_{m,n}(u,v) = \frac{\partial Q_{m,n}(u,v)}{\partial v} := \sum_{i=0}^{m} \sum_{j=0}^{n-1} \frac{\partial P_{i,j}}{\partial j} Be(u)_i^m(t) Be(v)_j^{n-1}(t)$$

Bezier surface normal: $n_{m,n}(u,v) := \frac{\partial Q_{m,n}(u,v)}{\partial u} \times \frac{\partial Q_{m,n}(u,v)}{\partial v}$