Prezentácia z formálnych jazykov a automatov

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Zadanie: Nech G je bezkontextová gramatika v *Chomského normálnom tvare* s k pravidlami očíslovanými $1, 2, ..., k$. Nech $L = \{wu | w \in L(G)$ a u je postupnosť čísel pravidiel použitých v ľavom krajnom odvodení w v $G\}$. Dokážte, že $L \in DSPACE(n^2)$.

Riešenie: Zostrojme teda, taký *(3-páskový) Deterministický Turingov Stroj* A, ktorý je $S(n^2)$ ohraničený a $L(A) = L$.

Konštrukcia A: Nech $G = (N, T, P, \sigma)$, $I = \{1, 2, ..., m\}$ a $P = \{P_k | k \in I\}$, potom $A = (K, \Sigma, \Gamma, \delta, q_0, F)$, kde :

$$
\begin{aligned}
K &= \{q_0, q_s, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_A\} \cup \{q_\alpha | C \to \alpha \in P\} \\
\Sigma &= T \cup I \\
\Gamma &= T \cup N \cup I \cup \{\mathbf{B}\} \\
F &= \{q_A\}
\end{aligned}
$$

Princíp: A bude pracovať na 3 páskach. Na Prvej má uložený vstup a na druhej a tretej postune simuluje ľavé krajné odvodenie v gramatike G.

1. Prejde terminálnim prefixom na prvej páske, najde prvú číslicu a na druhú pásku zapíše σ (q_0, q_s)

2. Kopíruje terminálny prefix z druhej pásky na tertiu. (q_1)

3. Podľa najľavejšieho neterminálu C na druhej páske a aktuálnej číslice k na prvej páske, zapíše α - pravú stranu pravidla $P_k = C \to \alpha$ (ak také existuje). (q_α)

4. Zvyšok druhej pásky prekopíruje na tretiu. (q_2)

5. Vráti sa na začiatok druhej a tretej pásky a skopíruje tretiu na druhú. (q_3, q_4)

6. Vráti sa na začiatok druhej a tretej pásky, prejde na ďalšiu číslicu na prvej páske a pokračuje v simulácii odvodenia. $(q_5, q_1, ...)$

7. Ak však prešiel na prvej páske na **B** , vráti sa začiatok prvej pásky a skontroluje znak po znaku vstupné slovo na prvej páske1 s aktuálnou vetnou formou na druhej páske, ak OK akceptuje. (q_6, q_7, q_A)

δ-funkcia

01	$\delta(q_0, x, \mathbf{B}, \mathbf{B})$	$=$	$(q_0, (x,1), (\mathbf{B},0), (\mathbf{B},0))$	$x \in T$
02	$\delta(q_0, k, \mathbf{B}, \mathbf{B})$	$=$	$(q_s, (k,0), (\mathbf{B},0), (\mathbf{B},0))$	$k \in I$
03	$\delta(q_s, k, \mathbf{B}, \mathbf{B})$	$=$	$(q_1, (k,0), (\sigma,0), (\mathbf{B},0))$	$k \in I,\ \sigma \in N$
04	$\delta(q_1, k, t, z)$	$=$	$(q_1, (k,0), (t,1), (t,1))$	$k \in I,\ t \in T, z \in N \cup T \cup \{\mathbf{B}\}$
05	$\delta(q_1, k, C, z)$	$=$	$(q_\alpha, (k,0), (C,0), (z,0))$	$k \in I,\ P_k = C \to \alpha$
06	$\delta(q_1, \mathbf{B}, z_1, z_2)$	$=$	$(q_6, (\mathbf{B},-1), (z_1,0), (z_2,0))$	$z_1, z_2 \in N \cup T \cup \{\mathbf{B}\}$
07	$\delta(q_{x\alpha}, k, C, z)$	$=$	$(q_\alpha, (k,0), (C,0), (x,1))$	$k \in I,\ C \in N,\ x \in N \cup T,\ z \in N \cup T \cup \{\mathbf{B}\}$
08	$\delta(q_\varepsilon, k, C, z)$	$=$	$(q_2, (k,0), (C,1), (z,0))$	$k \in I,\ C \in N,\ z \in N \cup T \cup \{\mathbf{B}\}$
09	$\delta(q_2, k, z_1, z_2)$	$=$	$(q_2, (k,0), (z_1,1), (z_1,1))$	$k \in I,\ z_1 \in N \cup T,\ z_2 \in N \cup T \cup \{\mathbf{B}\}$
10	$\delta(q_2, k, \mathbf{B}, z_2)$	$=$	$(q_3, (k,0), (\mathbf{B},-1), (z_2,-1))$	$k \in I,\ z_2 \in N \cup T \cup \{\mathbf{B}\}$
11	$\delta(q_3, k, z_1, z_2)$	$=$	$(q_3, (k,0), (z_1,-1), (z_2,-1))$	$k \in I,\ z_1, z_2 \in N \cup T$
12	$\delta(q_3, k, \mathbf{B}, z_2)$	$=$	$(q_3, (k,0), (\mathbf{B},0), (z_2,-1))$	$k \in I,\ z_2 \in N \cup T$
13	$\delta(q_3, k, \mathbf{B}, \mathbf{B})$	$=$	$(q_4, (k,0), (\mathbf{B},1), (\mathbf{B},1))$	$k \in I$
14	$\delta(q_4, k, z_1, z_2)$	$=$	$(q_4, (k,0), (z_2,1), (z_2,1))$	$k \in I,\ z_1 \in N \cup T \cup \{\mathbf{B}\},\ z_2 \in N \cup T$
15	$\delta(q_4, k, \mathbf{B}, \mathbf{B})$	$=$	$(q_5, (k,0), (\mathbf{B},-1), (\mathbf{B},-1))$	$k \in I$
16	$\delta(q_5, k, z_1, z_2)$	$=$	$(q_5, (k,0), (z_1,-1), (z_2,-1))$	$k \in I,\ z_1, z_2 \in N \cup T$
17	$\delta(q_5, k, \mathbf{B}, \mathbf{B})$	$=$	$(q_1, (k,1), (\mathbf{B},1), (\mathbf{B},1))$	$k \in I$
18	$\delta(q_6, z, z_1, z_2)$	$=$	$(q_6, (z,-1), (z_1,0), (z_2,0))$	$z \in \Sigma,\ z_1, z_2 \in N \cup T \cup \{\mathbf{B}\}$
19	$\delta(q_6, \mathbf{B}, z_1, z_2)$	$=$	$(q_7, (\mathbf{B},1), (z_1,0), (z_2,0))$	$z_1, z_2 \in N \cup T \cup \{\mathbf{B}\}$
20	$\delta(q_7, x, x, z)$	$=$	$(q_7, (x,1), (x,1), (z,0))$	$x \in T,\ z \in N \cup T \cup \{\mathbf{B}\}$
21	$\delta(q_7, k, \mathbf{B}, z)$	$=$	$(q_A, (k,0), (\mathbf{B},0), (z,0))$	$k \in I,\ z \in N \cup T \cup \{\mathbf{B}\}$

Dôkaz: Dokážme najprv, že takto navrhnutý *Deterministický Turingov Stroj* A akceptuje práve jazyk L, teda $L(A) = L$

$L(A) \subseteq L$**:** $w \in L(A) \stackrel{?}{\Longrightarrow} w \in L$. Vieme, že $w \in L(A) \Longrightarrow (g_0, (w,0), (\varepsilon,0), (\varepsilon,0)) \vdash^* (g_A, (v_1, l_1), (v_2, l_2), (v_3, l_3))$. Nech $w = uv\gamma \quad u \in T^*,\ v \in I^*,\ \gamma \in \Sigma^*,\ t \in T^*,$

$C \in N,\ \beta \in (N \cup T)^*$, potom platí:

- Indukciou na u: $(g_0,(w,0),(\varepsilon,0),(\varepsilon,0)) \vdash^* (g_0,(w,|u|),(\varepsilon,0),(\varepsilon,0))$
  1° $u=\varepsilon$ $\quad (g_0,(w,0),(\varepsilon,0),(\varepsilon,0)) \vdash^0 (g_0,(w,|u|),(\varepsilon,0),(\varepsilon,0))$
  2° $u=u_0x$ $\quad (g_0,(w,0),(\varepsilon,0),(\varepsilon,0)) \vdash_{IP}^{|u_0|} (g_0,(w,|u_0|),(\varepsilon,0),(\varepsilon,0)) \vdash_{(01)}$
  $(g_0,(w,|u|),(\varepsilon,0),(\varepsilon,0))$

- $(g_0,(w,|u|),(\varepsilon,0),(\varepsilon,0)) \vdash_{(02)} (g_s,(w,|u|),(\varepsilon,0),(\varepsilon,0))$

- $(g_s,(w,|u|),(\varepsilon,0),(\varepsilon,0)) \vdash_{(03)} (g_1,(w,|u|),(\sigma,0),(\varepsilon,0))$

- Indukciou na t: $(g_1,(w,|u|),(tC\beta,0),(\varepsilon,0)) \vdash^* (g_1,(w,|u|),(tC\beta,|t|),(t,|t|))$
  1° $t=\varepsilon$ $\quad (g_1,(w,|u|),(tC\beta,0),(\varepsilon,0)) \vdash^0 (g_1,(w,|u|),(tC\beta,|t|),(t,|t|))$
  2° $t=t_0x$ $\quad (g_1,(w,|u|),(tC\beta,0),(\varepsilon,0)) \vdash_{IP}^{|t_0|} (g_1,(w,|u|),(tC\beta,|t_0|),(t_0,|t_0|)) \vdash_{(04)}$
  $(g_1,(w,|u|),(tC\beta,|t|),(t,|t|))$

- $(g_1,(w,|u|),(tC\beta,|t|),(t,|t|)) \vdash_{(05)} (g_\alpha,(w,|u|),(tC\beta,|t|),(t,|t|))$
  $P_k = C \to \alpha$

- Indukciou na α: $(g_\alpha,(w,|u|),(tC\beta,|t|),(t,|t|)) \vdash^* (g_\varepsilon,(w,|u|),(tC\beta,|t|),(t\alpha,|t\alpha|))$
  1° $\alpha=\varepsilon$ $\quad (g_\alpha,(w,|u|),(tC\beta,|t|),(t,|t|)) \vdash^0 (g_\varepsilon,(w,|u|),(tC\beta,|t|),(t\alpha,|t\alpha|))$
  2° $\alpha=x\alpha_0$ $\quad (g_\alpha,(w,|u|),(tC\beta,|t|),(t,|t|)) \vdash_{(07)} (g_{\alpha_0},(w,|u|),(tC\beta,|t|),(tx,|tx|)) \vdash_{IP}^{|\alpha_0|}$
  $(g_\varepsilon,(w,|u|),(tC\beta,|t|),(t\alpha,|t\alpha|))$

- $(g_\varepsilon,(w,|u|),(tC\beta,|t|),(t\alpha,|t\alpha|)) \vdash_{(08)} (g_2,(w,|u|),(tC\beta,|tC|),(t\alpha,|t\alpha|))$

- Indukciou na β: $(g_2,(w,|u|),(tC\beta,|tC|),(t\alpha,|t\alpha|)) \vdash^* (g_2,(w,|u|),(tC\beta,|tC\beta|),(t\alpha\beta,|t\alpha\beta|))$
  1° $\beta=\varepsilon$ $\quad (g_2,(w,|u|),(tC\beta,|tC|),(t\alpha,|t\alpha|)) \vdash^0 (g_2,(w,|u|),(tC\beta,|tC|),(t\alpha\beta,|t\alpha\beta|))$
  2° $\beta=\beta_0x$ $\quad (g_2,(w,|u|),(tC\beta,|tC|),(t\alpha,|t\alpha|)) \vdash_{IP}^{|\beta_0|}$
  $(g_2,(w,|u|),(tC\beta,|tC\beta_0|),(t\alpha\beta_0,|t\alpha\beta_0|)) \vdash_{(09)}$
  $(g_2,(w,|u|),(tC\beta,|tC\beta|),(t\alpha\beta,|t\alpha\beta|))$

- $(g_2,(w,|u|),(tC\beta,|tC\beta|),(t\alpha\beta,|t\alpha\beta|)) \vdash_{(10)}$
  $(g_3,(w,|u|),(tC\beta,|tC\beta|-1),(t\alpha\beta,|t\alpha\beta|-1))$

- Indukciou na $tC\beta$[1] : $(g_3,(w,|u|),(tC\beta,|tC\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash^*$
  $(g_3,(w,|u|),(tC\beta,-1),(t\alpha\beta,-1))$
  1° $tC\beta=\varepsilon$ $\quad (g_3,(w,|u|),(tC\beta,|tC\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash^0$
  $(g_3,(w,|u|),(tC\beta,-1),(t\alpha\beta,-1))$
  2° $tC\beta=x\nu$ $\quad (g_3,(w,|u|),(tC\beta,|tC\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash_{(11),(12)}$
  $(g_3,(w,|u|),(tC\beta,|\nu|-1),(t\alpha\beta,|\nu|-1) \vdash_{IP}^{|\nu|}$
  $(g_3,(w,|u|),(tC\beta,-1),(t\alpha\beta,-1)$

- $(g_3,(w,|u|),(tC\beta,-1),(t\alpha\beta,-1)) \vdash_{(13)} (g_4,(w,|u|),(tC\beta,0),(t\alpha\beta,0))$

[1] Pre úplných formalistov. Treba si uvedomiť, že $|tC\beta| \leq |t\alpha\beta|$ (Chomského normálny tvar) a teda treba použiť dve indukcie najprv na $|tC\beta|$ znakov a potom druhú na 3.páske pre zvyšok

- Indukciou na $t\alpha\beta$: $(g_4,(w,|u|),(tC\beta,0),(t\alpha\beta,0)) \vdash^* (g_4,(w,|u|),(t\alpha\beta,|t\alpha\beta|),(t\alpha\beta,|t\alpha\beta|))$
  1° $t\alpha\beta=\varepsilon$ $\quad (g_4,(w,|u|),(tC\beta,0),(t\alpha\beta,0)) \vdash^0 (g_4,(w,|u|),(t\alpha\beta,|t\alpha\beta|),(t\alpha\beta,|t\alpha\beta|))$
  2° $t\alpha\beta=\nu x$ $\quad (g_4,(w,|u|),(tC\beta,0),(t\alpha\beta,0)) \vdash_{IP}^{|\nu|}$
  $(g_4,(w,|u|),(\nu,|\nu|),(\nu,|\nu|)) \vdash_{(14)}$
  $(g_4,(w,|u|),(t\alpha\beta,|t\alpha\beta|),(t\alpha\beta,|t\alpha\beta|))$

- $(g_4,(w,|u|),(t\alpha\beta,|t\alpha\beta|),(t\alpha\beta,|t\alpha\beta|)) \vdash_{(15)}$
  $(g_5,(w,|u|),(t\alpha\beta,|t\alpha\beta|-1),(t\alpha\beta,|t\alpha\beta|-1))$

- Indukciou na $t\alpha\beta$: $(g_5,(w,|u|),(t\alpha\beta,|t\alpha\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash^*$
  $(g_5,(w,|u|),(t\alpha\beta,-1),(t\alpha\beta,-1))$
  1° $t\alpha\beta=\varepsilon$ $\quad (g_5,(w,|u|),(t\alpha\beta,|t\alpha\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash^0$
  $(g_5,(w,|u|),(t\alpha\beta,-1),(t\alpha\beta,-1))$
  2° $t\alpha\beta=x\nu$ $\quad (g_5,(w,|u|),(t\alpha\beta,|t\alpha\beta|-1),(t\alpha\beta,|t\alpha\beta|-1)) \vdash_{(16)}$
  $(g_5,(w,|u|),(t\alpha\beta,|\nu|-1),(t\alpha\beta,|\nu|-1)) \vdash_{IP}^{|\nu|}$
  $(g_5,(w,|u|),(t\alpha\beta,-1),(t\alpha\beta,-1))$

- $(g_5,(w,|u|),(t\alpha\beta,-1),(t\alpha\beta,-1)) \vdash_{(17)} (g_1,(w,|u|+1),(t\alpha\beta,0),(t\alpha\beta,0))$

- Indukciou na v[2]: $(g_1,(w,|u|),(\gamma_1,0),(\gamma_2,0)) \vdash^* (g_1,(w,|uv|),(\gamma_1^{'},0),(\gamma_2^{'},0))$
  1° $v=\varepsilon$ $\quad (g_1,(w,|u|),(\gamma_1,0),(\gamma_2,0)) \vdash^0 (g_1,(w,|uv|),(\gamma_1^{'},0),(\gamma_2^{'},0))$
  2° $v=v_0x$ $\quad (g_1,(w,|u|),(\gamma_1,0),(\gamma_2,0)) \vdash_{IP}^{|v_0|}$
  $(g_1,(w,|uv_0|),(\gamma_1^{'},0),(\gamma_2^{'},0)) \vdash^* (g_1,(w,|uv|),(\gamma_1^{''},0),(\gamma_2^{''},0))$

- $(g_1,(w,|w|),(\gamma_1,0),(\gamma_2,0)) \vdash_{(06)} (g_6,(w,|w|-1),(\gamma_1,0),(\gamma_2,0))$[3]

- Indukciou na w: $(g_6,(w,|w|-1),(\gamma_1,0),(\gamma_2,0)) \vdash^* (g_6,(w,-1),(\gamma_1,0),(\gamma_2,0))$
  1° $w=\varepsilon$ $\quad (g_6,(w,|w|-1),(\gamma_1,0),(\gamma_2,0)) \vdash^0 (g_6,(w,-1),(\gamma_1,0),(\gamma_2,0))$
  2° $w=xw_0$ $\quad (g_6,(w,|w|-1),(\gamma_1,0),(\gamma_2,0)) \vdash_{(18)}$
  $(g_6,(w,|w_0|-1),(\gamma_1,0),(\gamma_2,0)) \vdash_{IP}^{|w_0|}$
  $(g_6,(w,-1),(\gamma_1,0),(\gamma_2,0))$

- $(g_6,(w,-1),(\gamma_1,0),(\gamma_2,0)) \vdash_{(19)} (g_7,(w,0),(\gamma_1,0),(\gamma_2,0))$

- Indukciou na u: $(g_7,(w,0),(\gamma_1,0),(\gamma_2,0)) \vdash^* (g_7,(w,|u|),(\gamma_1,|u|),(\gamma_2,0))$
  1° $u=\varepsilon$ $\quad (g_7,(w,0),(\gamma_1,0),(\gamma_2,0)) \vdash^0 (g_7,(w,|u|),(\gamma_1,|u|),(\gamma_2,0))$
  2° $u=u_0x$ $\quad (g_7,(w,0),(\gamma_1,0),(\gamma_2,0)) \vdash_{IP}^{|u_0|}$
  $(g_7,(w,|u_0|),(\gamma_1,|u_0|),(\gamma_2,0)) \vdash_{(20)}$
  $(g_7,(w,|u|),(\gamma_1,|u|),(\gamma_2,0))$

- $(g_7,(w,|u|),(\gamma_1,|u|),(\gamma_2,0)) \vdash_{(21)} (g_A,(w,|u|),(\gamma_1,|u|),(\gamma_2,0))$[4]

[2] Táto indukcia v sebe skrýva predošlé indukcie, najmä v 2°

[3] Vidíme, že $w=uv$ a $\beta=\varepsilon$, lebo inak by sa A zasekol.

[4] z δ-funkcie vyplýva, že $\gamma_1=u$, teda A akceptoval terminálne slovo odvodené v G ľavým krajným odvodením.

$L \subseteq L(A)$**:** $w \in L \overset{?}{\Longrightarrow} w \in L(A)$. Vieme, že $w \in L \Longrightarrow w = uv \;\; u \in T^*,\; v \in I^*$, kde v je postupnosť čísiel použitých pravidiel v ľavom krajnom odvodení w v G. Formálny postup tejto inklúzie je analogický, takže ho uvádzam. Treba si však uvedomiť, že teraz hľadáme výpočet A na w, teda argumentácia pri dokazovaní jednotlivých indukcií je jemne odlišná.

Dôkaz $S(n^2)$ **ohraničenosti** A**:** Uvažujme teda ľavé krajné odvodenie slova w v G[5]:

$$\sigma \equiv v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_n \equiv w$$

Dokžme MI nasledujúce tvrdenie:

$$\forall i (1 \leq i \leq n)(|v_i| \leq i)$$

$$
\begin{array}{ll}
1^\circ \; i = 1 & v_1 \equiv \sigma \Rightarrow |v_1| = |\sigma| = 1 \leq i \\
2^\circ \; i = k+1 & |v_k| \leq k \Rightarrow |v_i| \leq i \\
& \Big(v_k = tC\beta,\;\; t \in T^*,\; C \in N,\; \beta \in (N \cup T)^*,\; v_k \overset{(m)}{\Rightarrow} v_i, \\
& P_m = C \to \alpha\Big) \Longrightarrow v_i = \left\{ \begin{array}{lll} tAB\beta & \alpha = AB & \Longrightarrow |v_i| = |v_k| + 1 \\ tx\beta & \alpha = x & \Longrightarrow |v_i| = |v_k| \end{array} \right.
\end{array}
$$

Náš TS A simuluje ľavé krajné odvodenie v slova u v gramatike G (na vstupe $w = uv\beta$). Ak akceptuje w, tak $\beta = \varepsilon$ a teda z predošlého tvrdenia vyplýva, že A je dokonca $S(|w|)$ ohraničený. Ak však neakceptuje slovo w, dá sa podobne ukázať, že sa A zasekne, teda nezacyklí a teda nespotrebuje viac ako $S(|w|)$ pamäte.

[5] G je v Chomského normálnom tvare, tj. $P \subseteq N \times (T \cup NN)$