Fractals

Part 1 : Introduction & History

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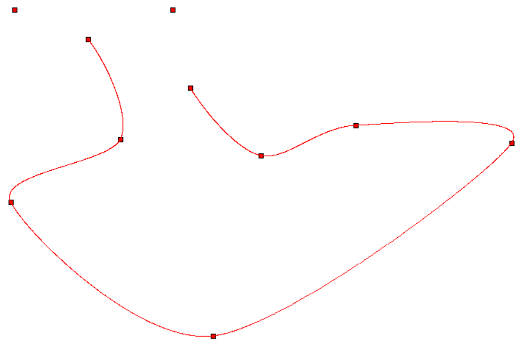
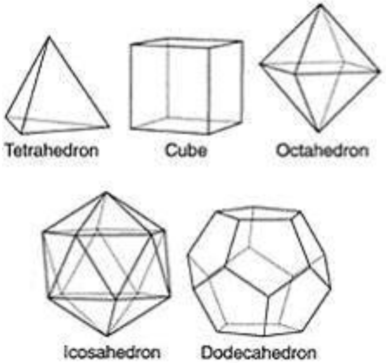
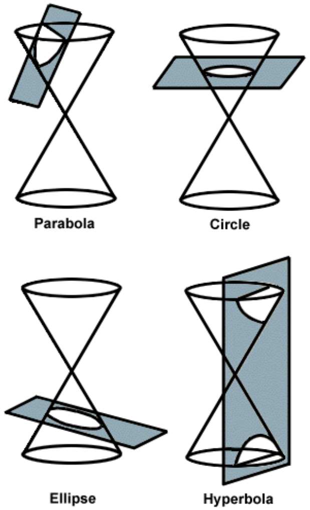
Classic geometry

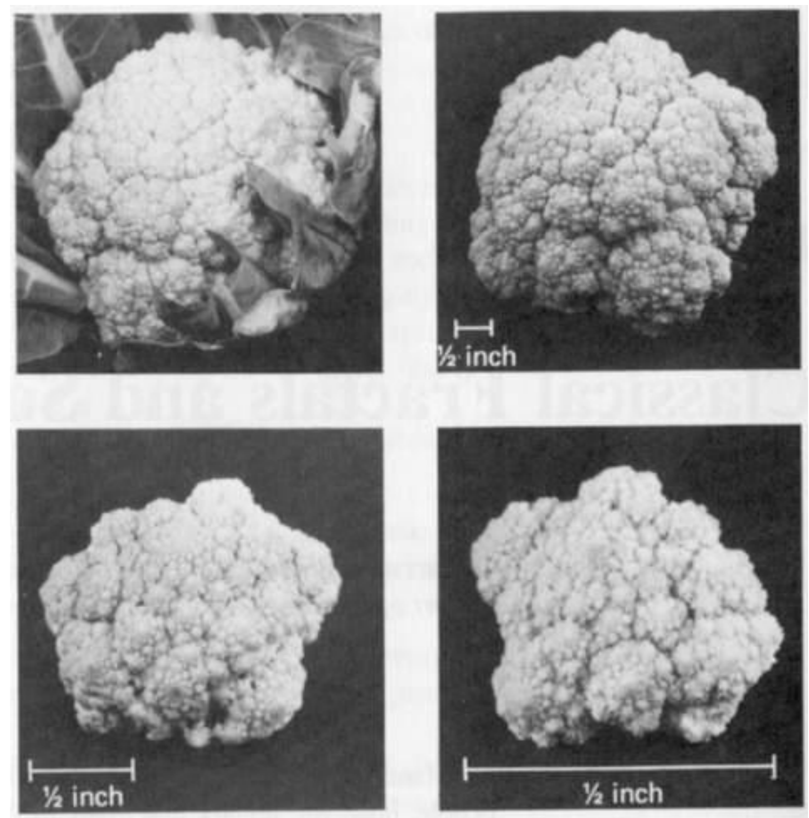
- Regular objects
- Line, circle, square, smooth curves
- Conics, smooth surfaces
- Hard to describe natural objects
- Simple when zooming







Nature

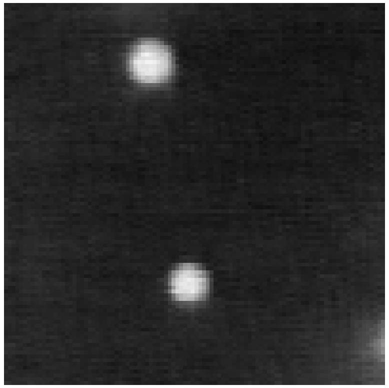


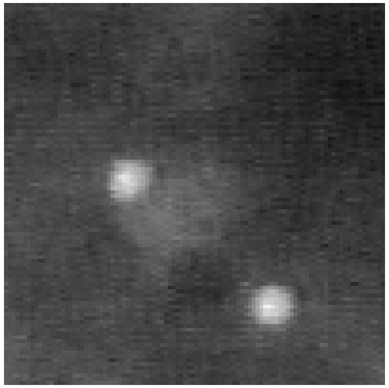
“Nature” geometry

- Irregular, fragmented objects, shapes
- Chaos inside
- Pretty to look
- Hard to describe
- Coastlines, trees, clouds, …
- Process of creation – repeated “operations”

Brownian motion

- Robert Brown – Scottish botanist
- random movement of a small particle when in a liquid or gas

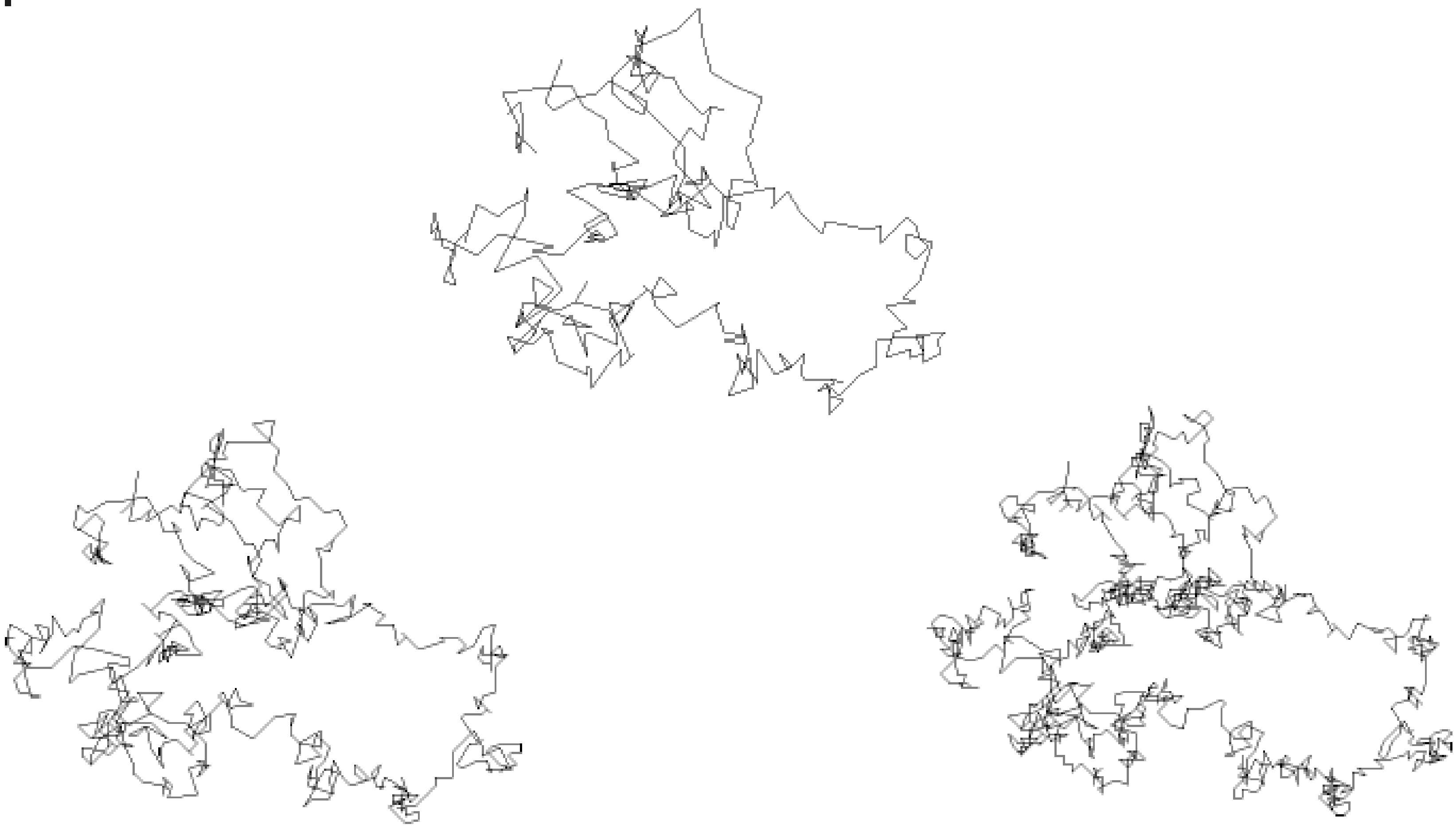




Brownian motion 2

- Jean Perrin
- tried to measure the velocity which is the derivative of the particle's position
- *"varies in the wildest way in magnitude and direction, and does not tend to a limit*"
- *"nature contains of non-differentiable as well as differentiable processes"*

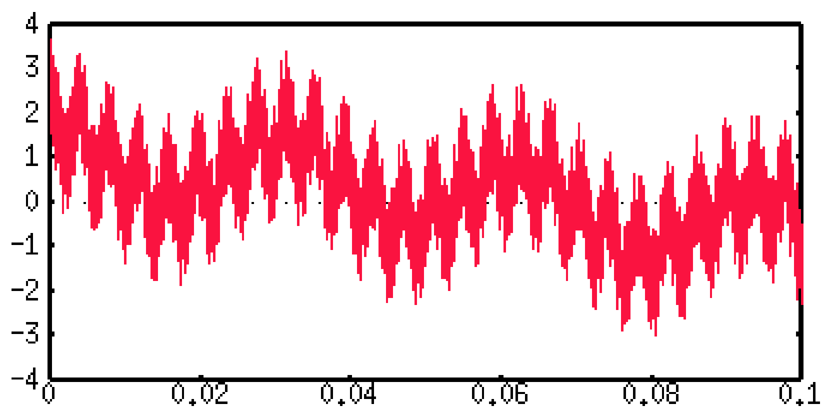
Brownian motion 3



Mathematical monsters

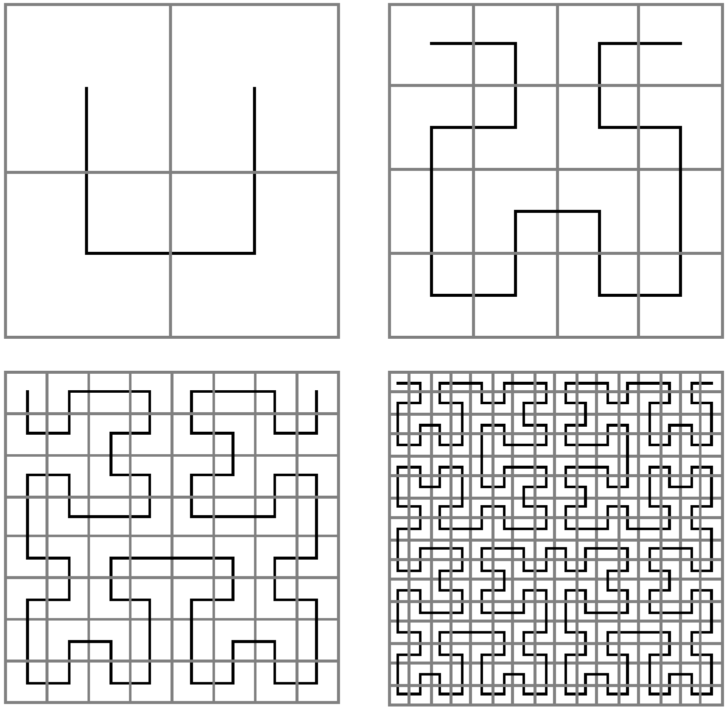
- Not differentiable functions
- Beginning of previous century
- Deplorable evil, pathological monster
- Karl Weierstrass

$$C(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$



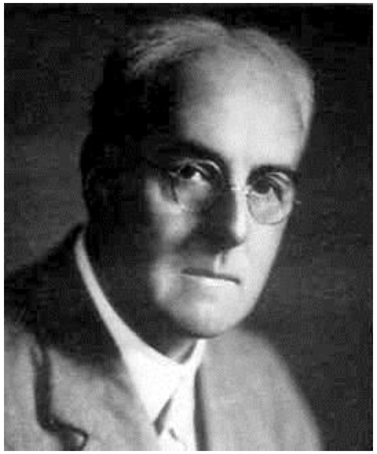
Mathematical monsters 2

- David Hilbert
- Space Filling Curve



Length of coastline

- What is the length of country's coastline?
- L.F. Richardson – Corsica
- B. Mandelbrot – Great Britain
- Spain-Portugal: 987-1214 km
- Netherlands-Belgium: 380-449 km
- 1cm:100km – 1cm:1km



Richardson's method

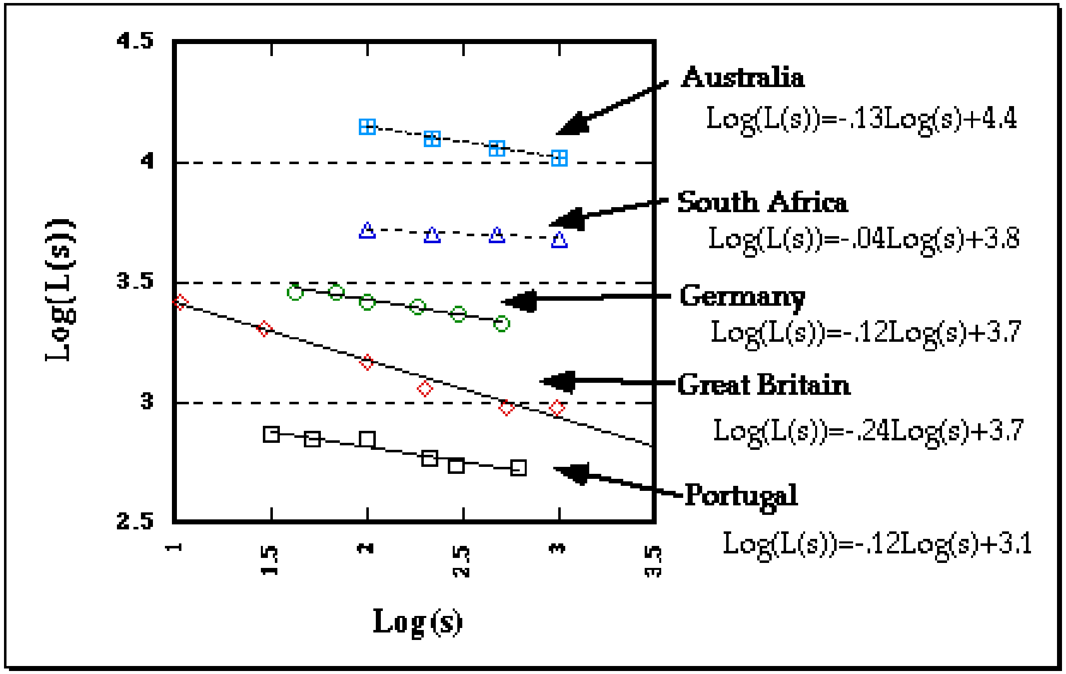
- Given ruler with constant length
- Measure length using ruler
- Using smaller ruler we get more details
- Length is increasing
- Infinite length
- Impossible to measure exactly

Richardson’s length

- While measuring coastline of Corsica
- $L(s) \approx Hs^{1-D}$, $N(s)s^{D} = H$
- L(s) – length of coastline measured using ruler with length s
- H - constant
- s – length of ruler
- D – constant (Richardson’s)
- N – number of pieces

Richardson's length 2

- Plot of log/log
- Mandelbrot: log[L(s)] = (1-D)log(s) + b



Length of curve

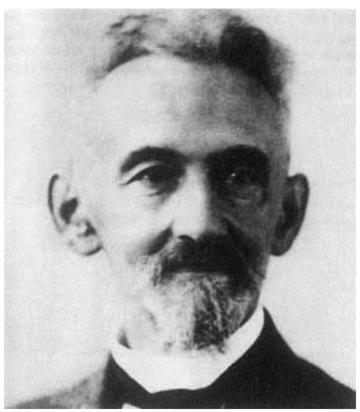
- Used approximation with open polygon
- Limit to infinity
- Length of given curve:

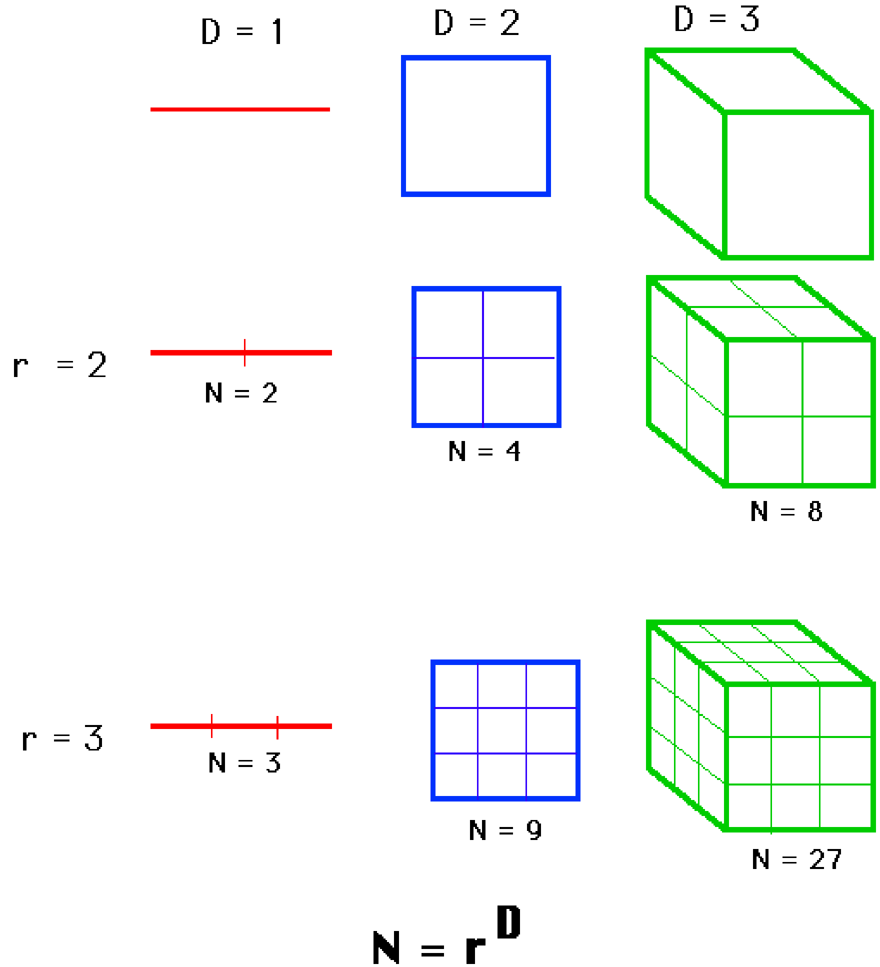
$$L(f) = \int \sqrt{1 + [f^{'}(t)]^2}\, dt$$

- What if there's no derivation?
- What if there's no exact function?

Topological dimension

- Integer
- F. Hausdorff
- L. E. J. Brouwer





Fractal dimension

- Based on Hausdorff-Besicovitch dimension in metric space
- *s* - scaling factor
- *N(s)* - number of coverage sets at factor *s*

$$N(s) \approx s^{-D} \qquad \mathit{for} \qquad s \to 0^{+}$$

$$D = -\lim_{s \to 0^{+}} \frac{\log N(s)}{\log s}$$

Fractal dimension 2

- Non-integer H-B dimension = fractal dimension
- Sometimes hard to compute it analytically
- Statistical methods
- Clipper method, box counting method, mass-radius method

Similarity dimension

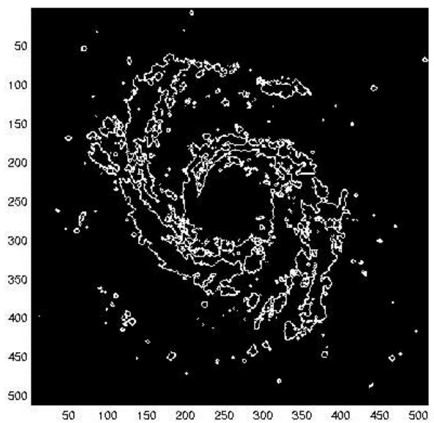
- Self-similarity fractals
- Easy to compute
- Analytical expression
- Formula

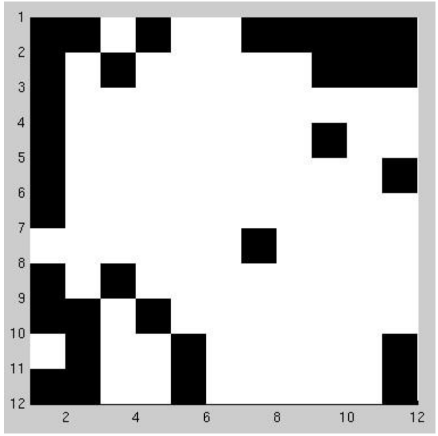
$$D = \frac{\log L(s)}{\log(1/s)}$$

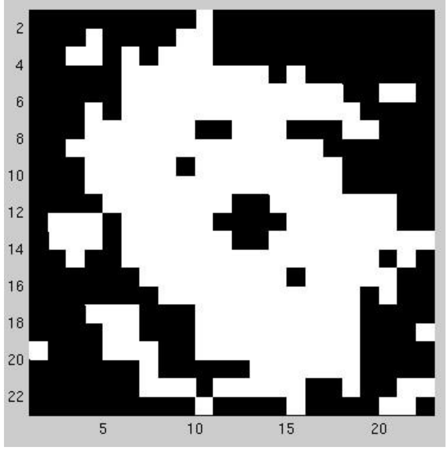
Box counting method

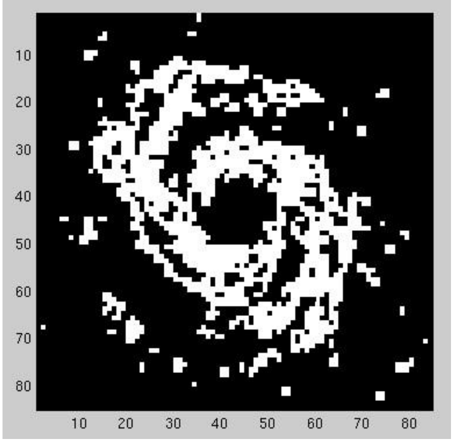
- Divide measuring space into boxes
- Count number of boxes containing set
- Do several measurements with different box sizes
- Make log N(s)/log(1/s) graph
- Slope of approximation line is fractal dimension

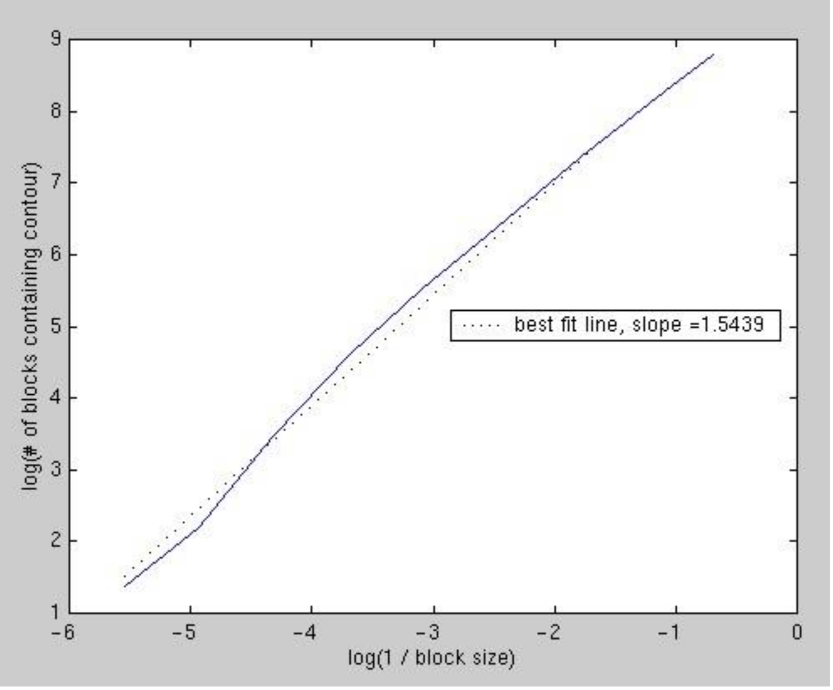
Box counting method 2











Fractals - definition

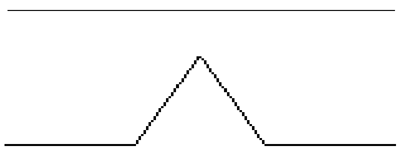
- Benoit Mandelbrot
- Fractal dimension > topological dimension
- Other definitions
- Based on fractal properties
- Fractus = to break, irregular

Fractals - properties

- Self-similarity, Self-affinity
- Deterministic
- Non-deterministic
- Extended Euclidean geometry
- Infinity
- Scaling independent
- Modeling, approximating natural objects

Koch's curve

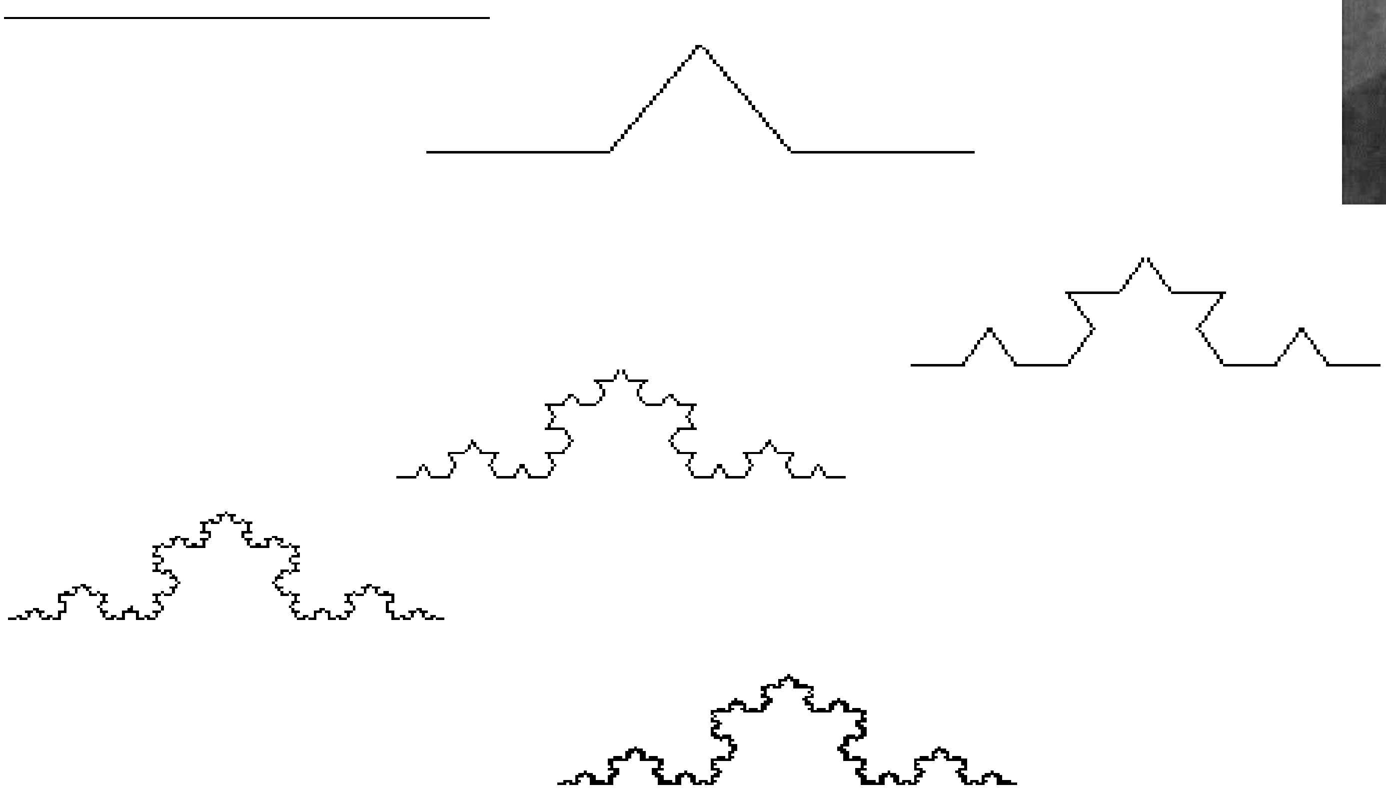
- Niels Fabian Helge von Koch
- Initiator: abscissa E
- Generator: 4 abscissas



- Algorithm: in E_{i-1} replace all abscissas with generator and you have E_i
- Koch's curve:

$$K = \lim_{n->\infty} E_n$$

Koch's curve 2



Koch's curve 3

- Length of K:

$$L(E_0) = l$$

$$L(E_1) = \frac{4}{3} l$$

$$L(E_i) = \frac{4}{3} L(E_{i-1})$$

$$L(E_i) = \left(\frac{4}{3}\right)^i l$$

$$L(K) = \infty$$

Koch's curve 4

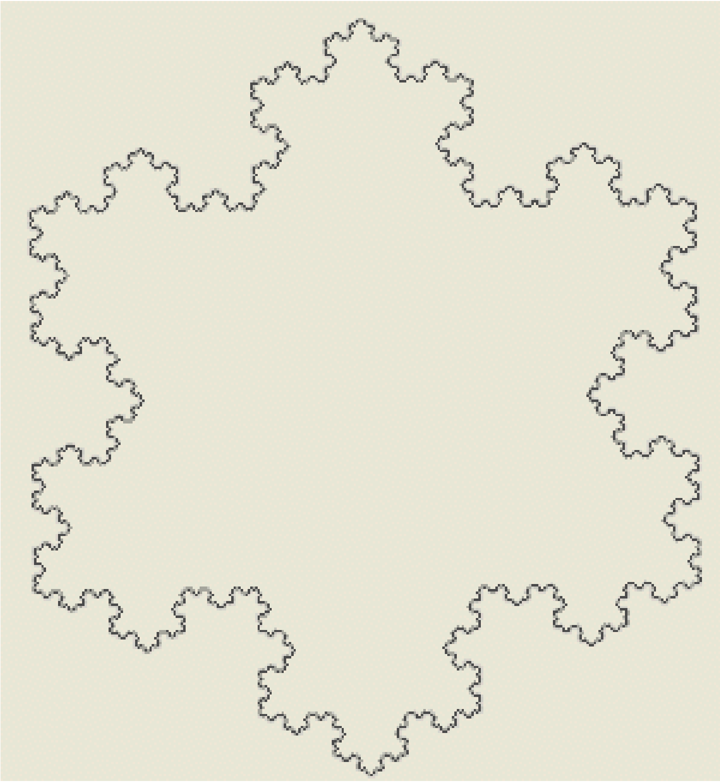
- Self-similarity dimension – easy to find
- For each (i-th) step $s = 1/3^i$, $N(s)=4^i$
- $D=\log N(s)/\log(1/s)=\log 4/\log 3$
- D = 1,261859….
- Statistically is near this value
- More than curve, less than plane
- Koch = Corsica, Great Britain

Koch's snowflake

- Measure inside snowflake (area):

$$S = \frac{2\sqrt{3}}{5} a^2$$

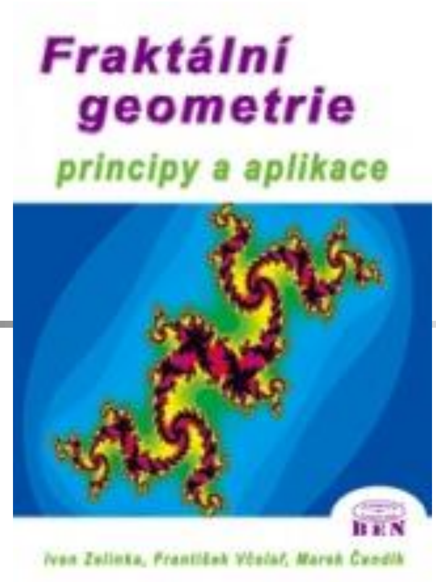
- Infinite length
- Zero curve area
- Finite space

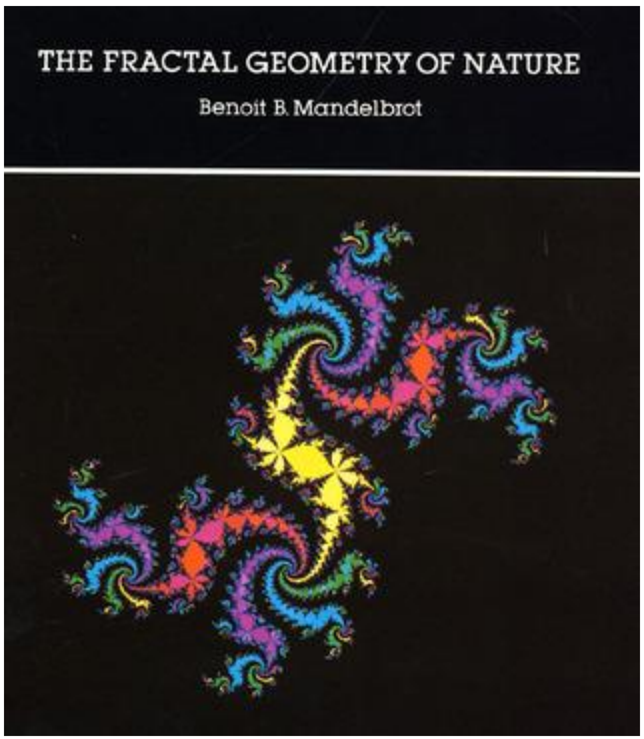


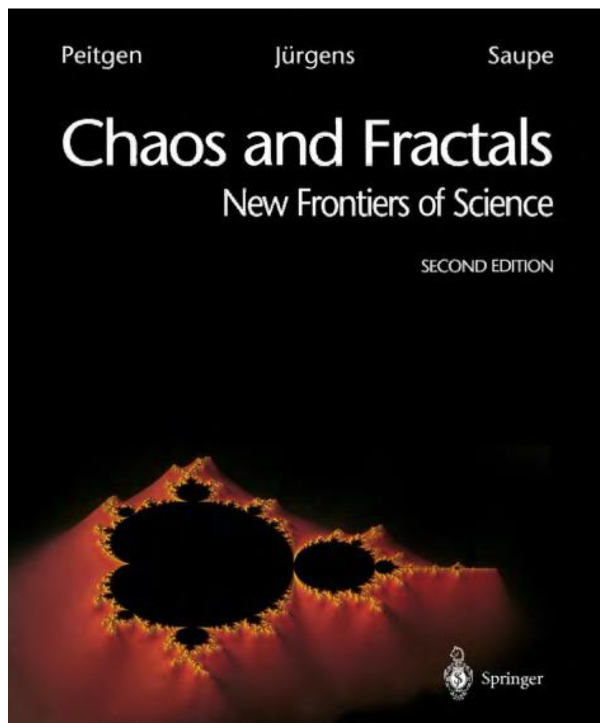
Generalized Koch's curve

- Can generate curves with different dimensions
- New parameter *a*
- For $a = 60^{\circ}$ we have standard Koch's curve
- Dimension: $$D = \frac{\log 4}{\log(2 + 2 * \cos\alpha)}$$

Literature


Fraktální geometrie
principy a aplikace
BEN
Ivan Zelinka, František Včelař, Marek Čandík


THE FRACTAL GEOMETRY OF NATURE
Benoit B. Mandelbrot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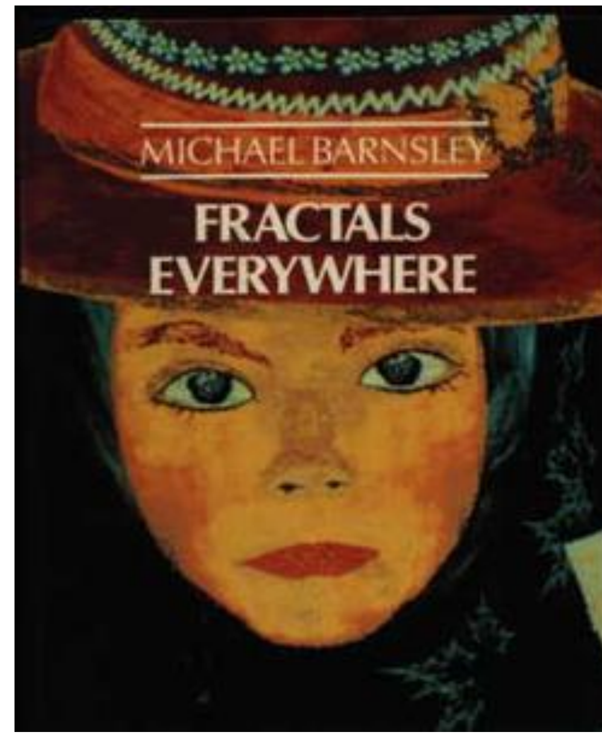
MICHAEL BARNSLEY
FRACTALS EVERYWHERE

Videos

- **Hunting the Hidden Dimension**
  - http://www.youtube.com/watch?v=ZbK92bRW2lQ
- **Clouds Are Not Spheres**
  - http://www.youtube.com/watch?v=Y9CFZbgJ94I
- **Fractals - The Colors Of Infinity**
  - http://www.youtube.com/watch?v=Lk6QU94xAb8

End

End of Part 1