Fractals

Part 3 : Mathematical background

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Space

- Space = set
- Is there relation between elements of this set ?
- Topology = how
- Geometry = where
- Relation = function of two arguments = metric

Metric space

- X – set (space)
- $d: X \times X \rightarrow R$
- $d(x,y) = d(y,x)$ (*symmetry*)
- $d(x,y) >= 0$
- $d(x,y) = 0 \iff x = y$
- $d(x,y) <= d(x,z) + d(y,z)$ (*triangle inequality*)

Metric spaces

- (X,d) – metric space
- Open ball:

  $$B(x, r) = \{y \text{ in } X : d(x,y) < r\}$$

- Circle:

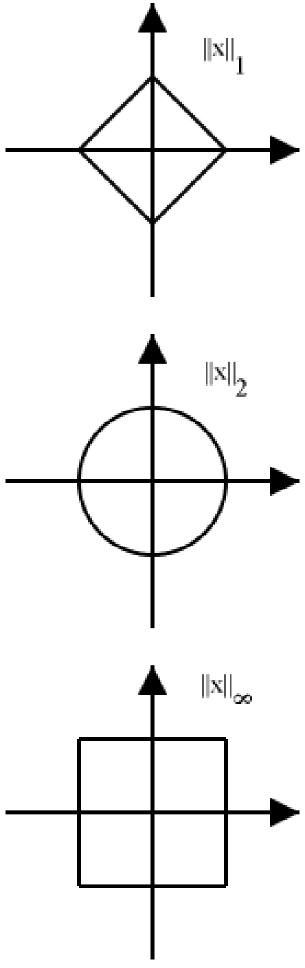
  $$C(x, r) = \{y \text{ in } X : d(x,y) = r\}$$

- r = radius

Metric spaces – examples

- Trivial: d(x,y)=0 if x=y else 1
- (R,d); d(x,y) = |x-y|
- Euclidean space (R^n,d)
- Manhattan (R^2,d_1)
- Generalized (R^n,d_p)
- Spherical space

$$d = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \qquad d_p = \sqrt[p]{\sum_{i=1}^{n}(x_i - y_i)^p}$$



Generalized spaces

- Topological space
- Vector space – vector addition, scalar multiplication
- Hilbert space – vector space with inner product
- Banach space – vector space with norm
- Hausdorff space

Limits

- Convergent sequence x_n have limit s
- For $\varepsilon>0$ exists $N>0$ so that $d(x_n,x)<\varepsilon$, for each $n>N$
- Notation: $x = \lim_{n\to\infty} x_n$
- Cauchy sequence: For $\varepsilon>0$ exists $N>0$ so that $d(x_n,x_m)<\varepsilon$, for each $n,m>N$
- Cauchy = knowing conv. without limit

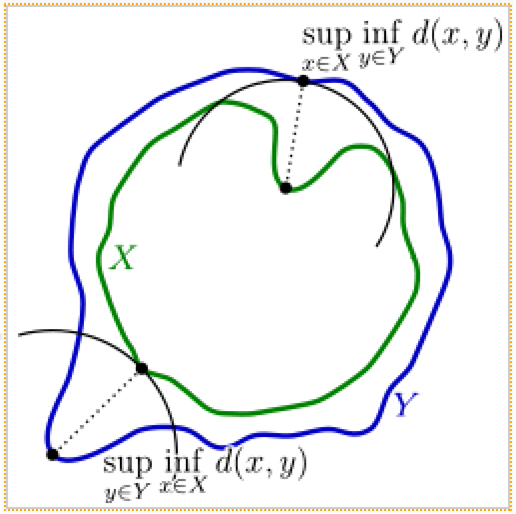
Complete & compact

- Complete space: if each Cauchy sequence has limit in space X
- Compact set: from any sequence of elements of S, a subsequence can always be extracted which tends to some limit element x of S
- Limit point: limit of any sequence
- Prefect set: S = limit points
- Closed set: S consists all limit points

Complete & compact 2

- Compact <=> is bounded and closed
- Cantor set is perfect
- Set of rational numbers is not complete, compact
- Euclid space with euclidean metrics is complete

More metrics



- In complete metric spaces
- d(x,A) = inf{d(x,y);y in A}; A is set
- ? d(A,B) = sup{d(x,B);x in A}
- d(A,B) <> d(B,A) – not a metric
- Solution: Hausdorff metric

  h(A,B) = max{d(A,B),d(B,A)}

Hausdorff space

- X – complete metric space
- H(X) – set of all non-empty compact subsets of X
- “Subset of set of all subsets”
- We need metric for compactness
- (H(X),h) –Hausdorff space

Hausdorff space 2

- h(A,B) is small for similar sets
- A_ε ={x | d(x,y) < ε for some y in A} - ε - collar
- h(A,B) = inf{ ε | A in B_ε & B in A_ε}
- If A,B are two points => h(A,B)=d(A,B)
- Many fractals “live” in (H(X),h)

Hausdorff measure

- diam(A) = sup{d(x,y);x,y in U}
- open cover of A: $A \subset \bigcup_{i=1}^{\infty} U_i$
- s-dimensional Hausdorff measure:

$$h_{\varepsilon}^{s}(A) = \inf\{\sum_{i=1}^{\infty} diam(U_i)^s\}; diam(U_i) < \varepsilon$$

$$h^{s}(A) = \lim_{\varepsilon \to 0} h_{\varepsilon}^{s}(A)$$

- Can be used for length of curve, area of surface, …

Hausdorff dimension

- For any set A there exist number D(A)
- For s<D(A) is measure infinite
- For s>D(A) is measure 0
- D(A) – Hausdorff dimension
- D(A) can be infinite, 0, any positive real number

Transformations

- In metric space X
- Transformation on X is function f:X -> X
- If S is subset of X, $f(S)=\{f(x) \mid x \text{ in } S\}$
- Forward iterations $x, f(x), f(f(x)), .., f^n(x), ..$
- Orbit, Infinitely process
- We need to measure transformations

Transformations 2

- Affine: preserving collinearity and ratios of distances – scaling, translation, rotation, shearing
- Affine: $F(x)=Ax+b$
- Polynomial
- Linear fractional (Mobius)

Contraction transformation

- Contraction: $d(f(x),f(y)) <= Cd(x,y)$ for each x,y in X
- C in <0,1)
- C depends on metric
- Contraction ($C < 1$), expansion ($C > 1$), symmetry ($C = 1$)
- Contraction is always continuous

Fixed point theorem

- Banach
- (X,d) complete metric space
- f – contraction
- There exists unique fixed point x_f: $f(x_f)=x_f$
- For each x in X sequence $\{f^n(x)\}$ converges to x_f

Fixed point theorem 2

- Simple proof
- Speed of conversion

$$d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0).$$

- Base for many fractals
- Generalized for sets
- Other fixed point theorems

Transformations in HS

- f(B) = {f(x) : x in B}, for B in H(X)
- If f is contraction in X, then is in H(X)
- Contractions {f1, f2, ..., fN} with factors {s1, s2, ..., sN}.
- W: H(X) -> H(X) $W(B) = \bigcup_{n=1}^{N} f_n(B) \quad \forall B \in H(X)$
- W is contraction with factor
  s = max {s1, s2, ..., sN}

End

End of Part 3