Fractals

Part 6 : Julia and Mandelbrot sets, ...

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Problem of initial points

- Newton method for computing root of function numerically
- Computing using iterations

$$y_{i+1} = y_i - \frac{f(y_i)}{f^{'}(y_i)}$$

- For given root, which initial points lead to this root?

Example

- Equation: $z^3 - 1$
- 3 roots in complex plane
- Newton method, sequence

$$y_{n+1} = y_n - \frac{y_n^3 - 1}{2y_n^2}$$

- What is basin of attraction?
- What are boundaries of 3 basins?

Pixel game

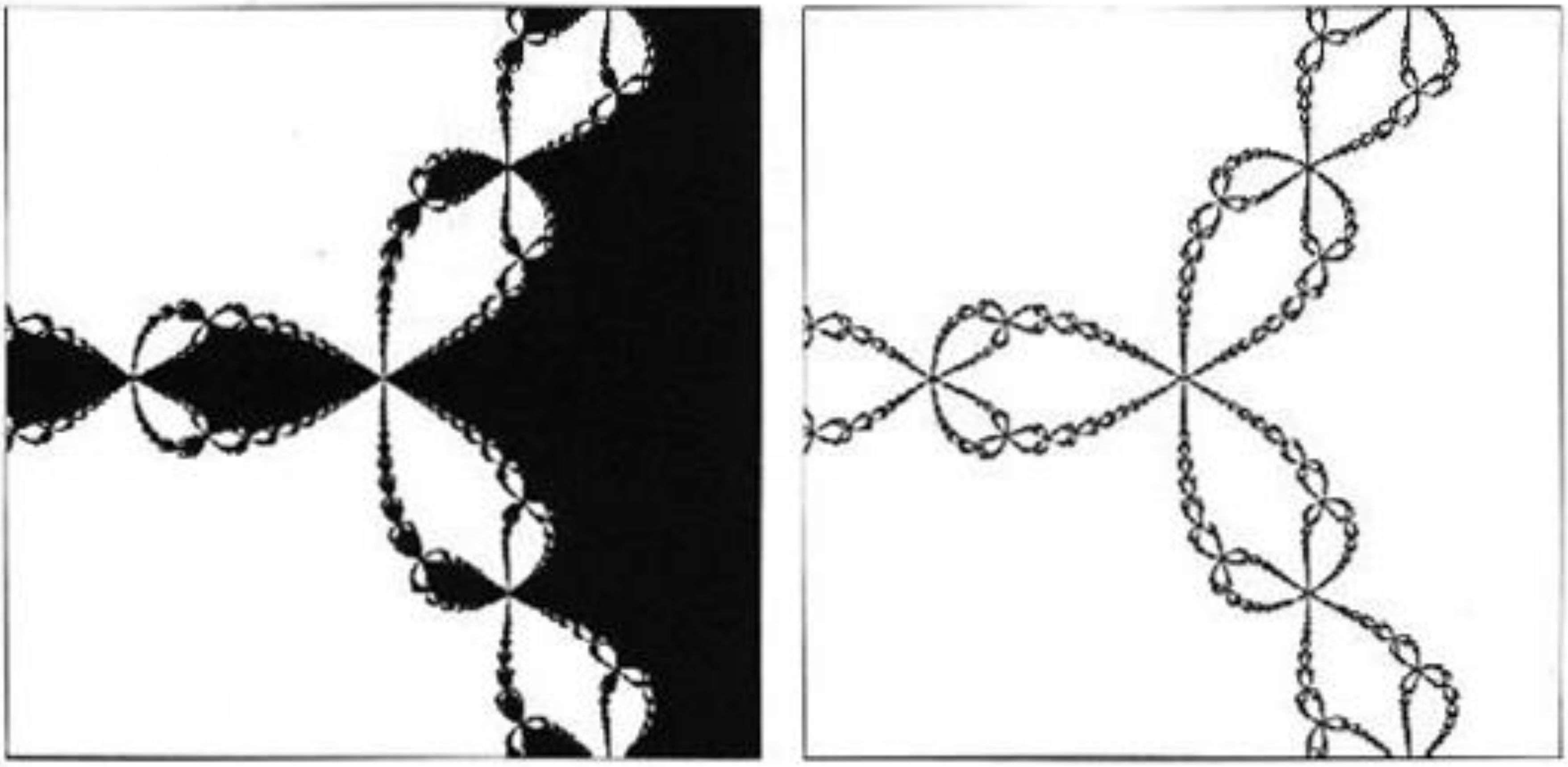
- Starting with discrete board
- Picked initial point (square)
- Two cases: periodic and fixed
- Basin of attraction of fixed square = set of initial squares that lead to initial square
- Source of steps?

Pixel game 2

L	K 2	K 3	K 3	K 4	K 4	I 5	I 6	I 7	I 8	I 8	I 9
K	K 3	K 3	X	K 3	I 3	H 4	H 5	G 7	H 8	H 9	H 9
I	I 3	I 3	K 4	K 3	I 2	G 3	F 5	F 7	F 8	G 9	G 10
H	H 3	I 4	K 5	L 3	K 1	D 2	B 5	D 7	F 9	F 9	G 10
G	G 4	H 5	K 7	K 6	L 2	A 2	C 7	C 10	E 10	F 10	F 10
F	F 4	F 6	F 9	F 10	F 11	F 11	F 11	F 11	F 11	X	F 10
E	E 4	D 5	B 7	B 6	A 2	L 2	I 7	I 10	G 10	F 10	F 10
D	D 3	C 4	B 5	A 3	B 1	H 2	K 5	H 7	F 9	F 9	E 10
C	C 3	C 3	B 4	B 3	C 2	E 3	F 5	F 7	F 8	E 9	E 10
B	B 3	B 3	X	B 3	C 3	D 4	D 5	E 7	D 8	D 9	D 9
A	B 2	B 3	B 3	B 4	B 4	C 5	C 6	C 7	C 8	C 8	C 9
	1	2	3	4	5	6	7	8	9	10	11

2	1	1	2	2	5	6	4	4	4	4
1	1		1	3	3	3	5	6	4	4
3	3	2	1	4	5	3	3	3	3	2
5	2	4	2	2	3	4	5	3	3	2
5	3	4	4	2	2	4	4	2	1	1
2	3	3	1	2	2	2	2	2		1
5	3	4	4	2	2	4	4	2	1	1
5	2	4	2	2	3	4	5	3	3	2
3	3	2	1	4	5	3	3	3	3	2
1	1		1	3	3	3	5	6	4	4
2	1	1	2	2	5	6	4	4	4	4

Newton fractal



Complex numbers

- 3 types of notation:

$$a+bi; r(\cos(\varphi)+i\sin(\varphi)); r.e^{i\varphi}$$

- Simple addition, multiplication
- Operations like with real numbers
- Square roots
- Equations

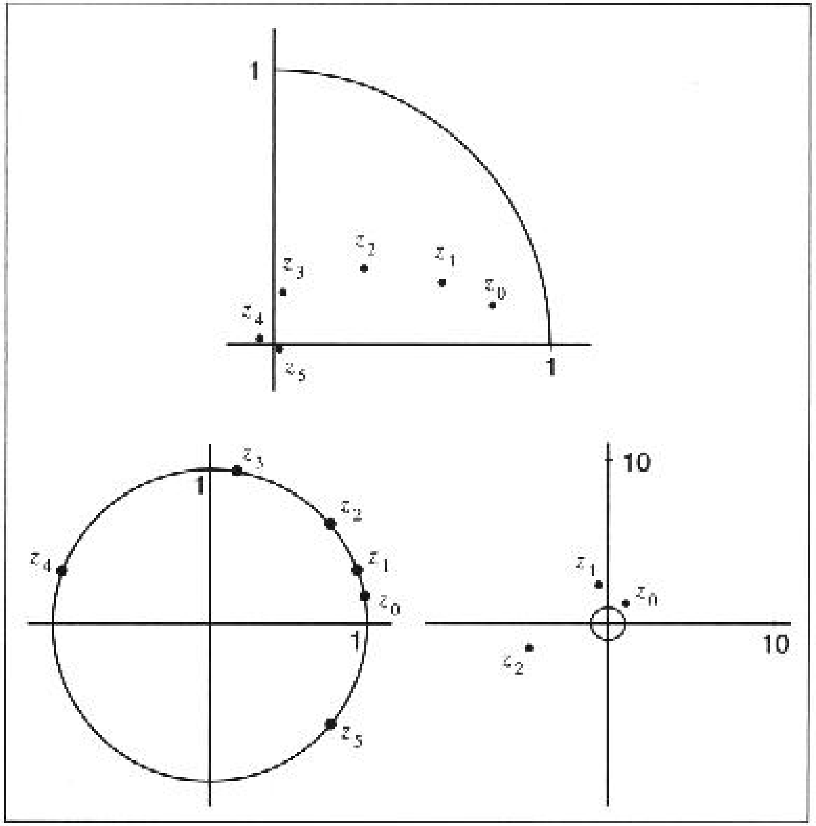
Prisoners, escapees

- Observe $z \rightarrow z^2$
- Infinite iterations = orbits
- For points in unit circle we have prisoners
- Else we have escapees
- Escape set E, prisoner set P
- Boundary between E,P = Julia set

Prisoners, escapees 2

- Invariant under iteration

	length	angle	length	angle	length	angle
z	0.8	10°	1.0	10°	1.5	50°
z^2	0.64	20°	1.0	20°	2.25	100°
z^4	0.4096	40°	1.0	40°	5.06	200°
z^8	0.1678	80°	1.0	80°	25.63	40°
z^{16}	0.0281	160°	1.0	160°	656.90	80°
z^{32}	0.0008	320°	1.0	320°	431439.89	160°



Extending

- $z^2 + c$

$$z_{n+1} = z_n^2 + c$$

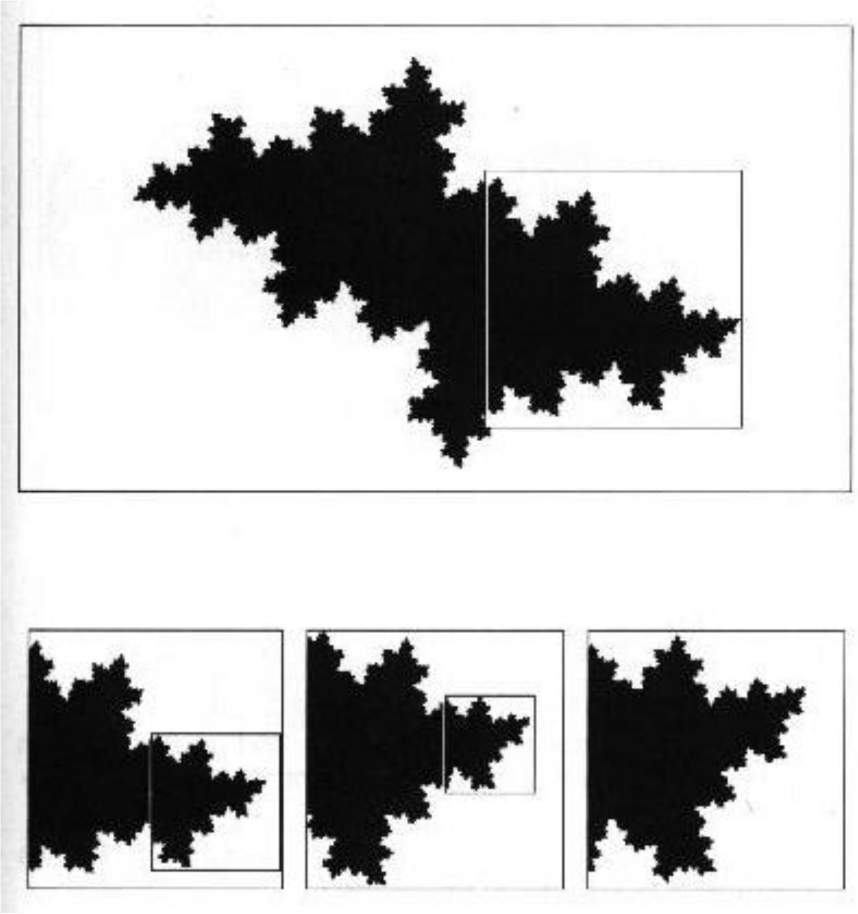
- Julia set
- Shape?

	Orbit 1		Orbit 2		Orbit 3	
	x	y	x	y	x	y
z_0	1.00	0.00	0.50	0.25	0.00	0.88
z_1	0.50	0.50	-0.31	0.75	-1.27	0.50
z_2	-0.50	1.00	-0.96	0.03	0.87	-0.77
z_3	-1.25	-0.50	0.43	0.44	-0.34	-0.85
z_4	0.81	1.75	-0.51	0.88	-1.12	1.07
z_5	-2.90	3.34	-1.01	-0.39	-0.41	-1.90
z_6	-3.26	-18.91	0.37	1.30	-3.93	2.04
z_7	-347.46	123.68	-2.04	1.46	10.79	-15.52
z_8			1.53	-5.46	-124.77	-334.49
z_9			-28.01	-16.27		

	Orbit 1		Orbit 2		Orbit 3	
	x	y	x	y	x	y
z_0	0.000	0.000	0.500	-0.250	-0.250	0.500
z_1	-0.500	0.500	-0.313	0.250	-0.688	0.250
z_2	-0.500	0.000	-0.465	0.344	-0.090	0.156
z_3	-0.250	0.500	-0.402	0.180	-0.516	0.472
z_4	-0.688	0.250	-0.371	0.355	-0.456	0.013
z_5	-0.090	0.156	-0.488	0.237	-0.292	0.488
z_{100}	-0.473	0.291	-0.393	0.290	-0.438	0.217
z_{200}	-0.394	0.279	-0.411	0.271	-0.409	0.290
z_{300}	-0.411	0.273	-0.409	0.276	-0.407	0.272
z_{400}	-0.408	0.276	-0.409	0.275	-0.409	0.276
z_{500}	-0.409	0.275	-0.409	0.275	-0.409	0.275

Julia set

c=-0.5+0.5i



Threshold radius

- When iteration leaves this radius, point is escaping
- r(c)=max(|c|,2)
- Using for visualization
- Easy proof

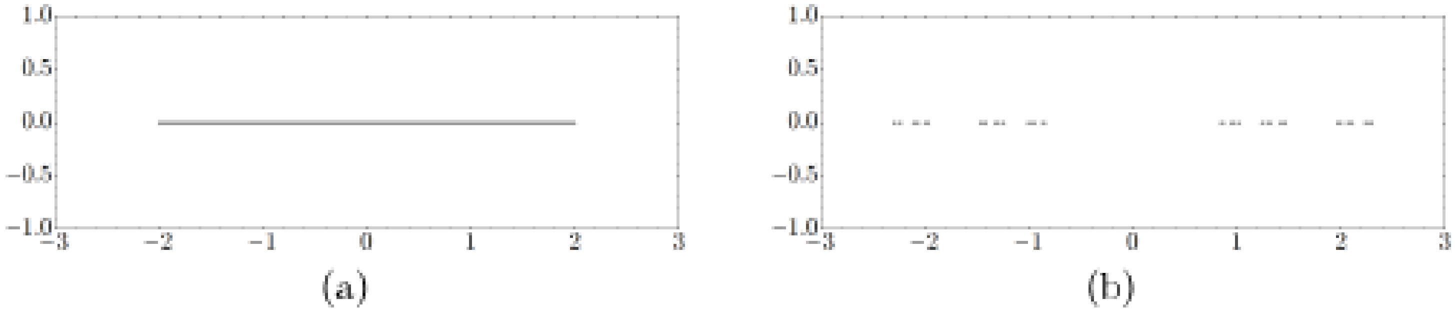


Figure 3: (a) Filled Julia set for $z^2 - 2$. (b) Filled Julia set for $z^2 - 3$.

Encirclement

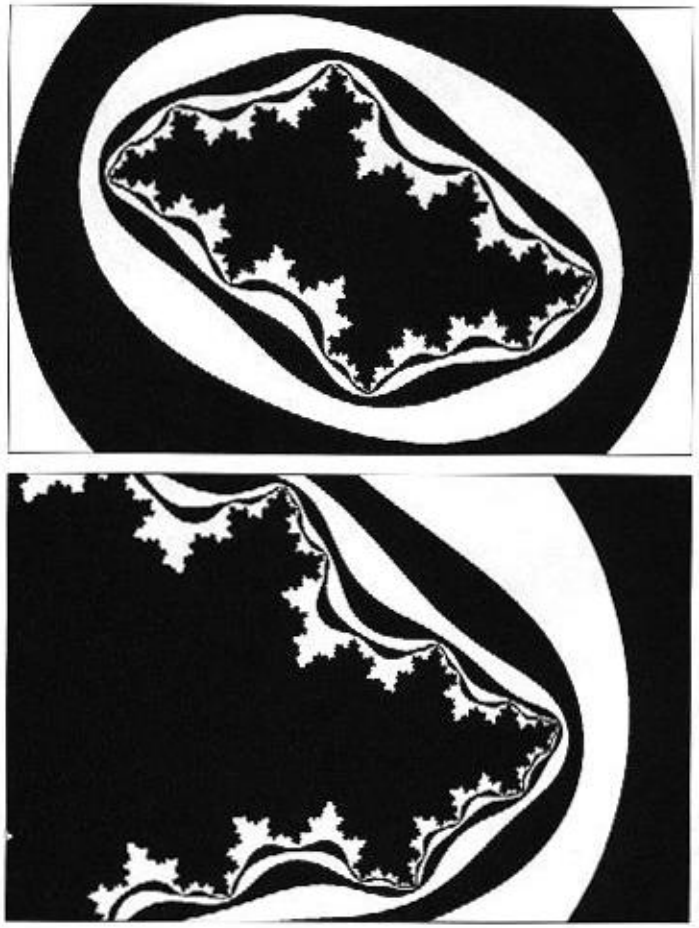
- Generalized threshold circle for any iteration step

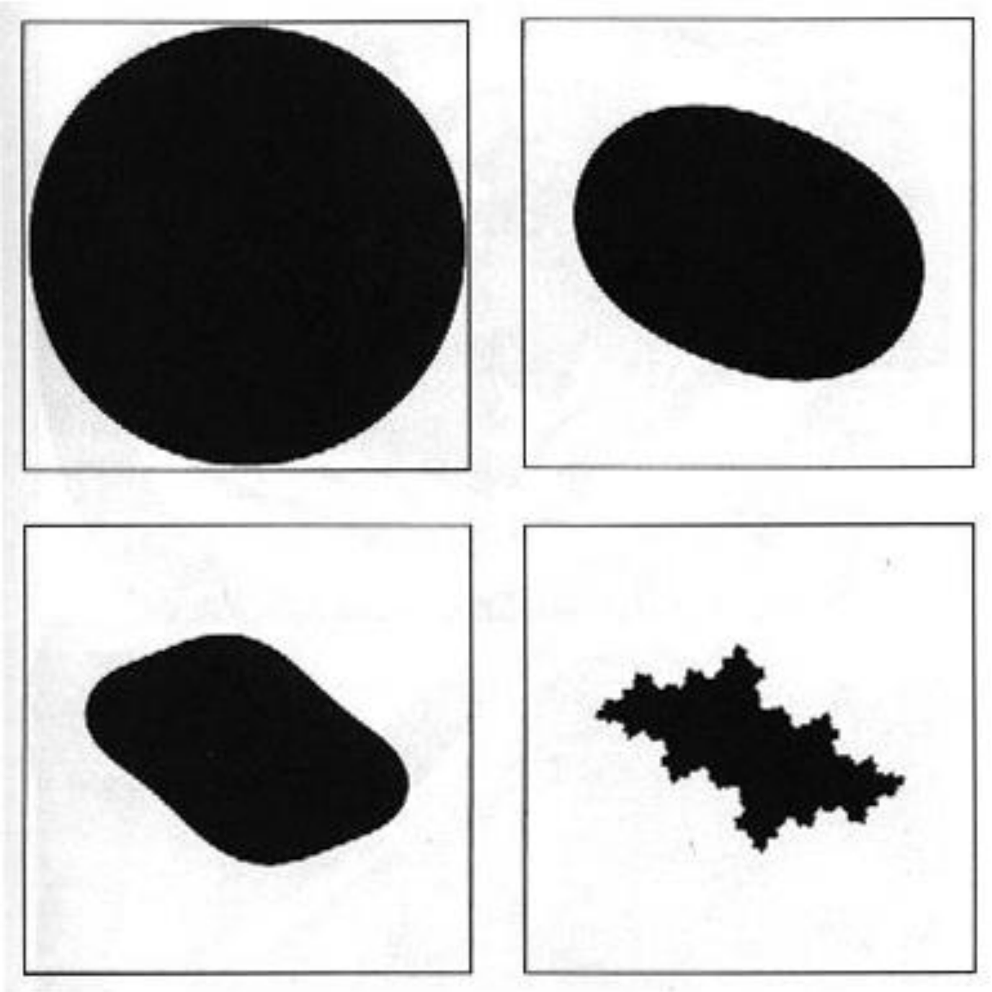
$$Q_c^{(-k)} = \{z_0; |z_k| \le r(c)\}; k = 0,1,.....$$

$$\lim_{k\to\infty} Q_c^{(-k)} = P_c$$

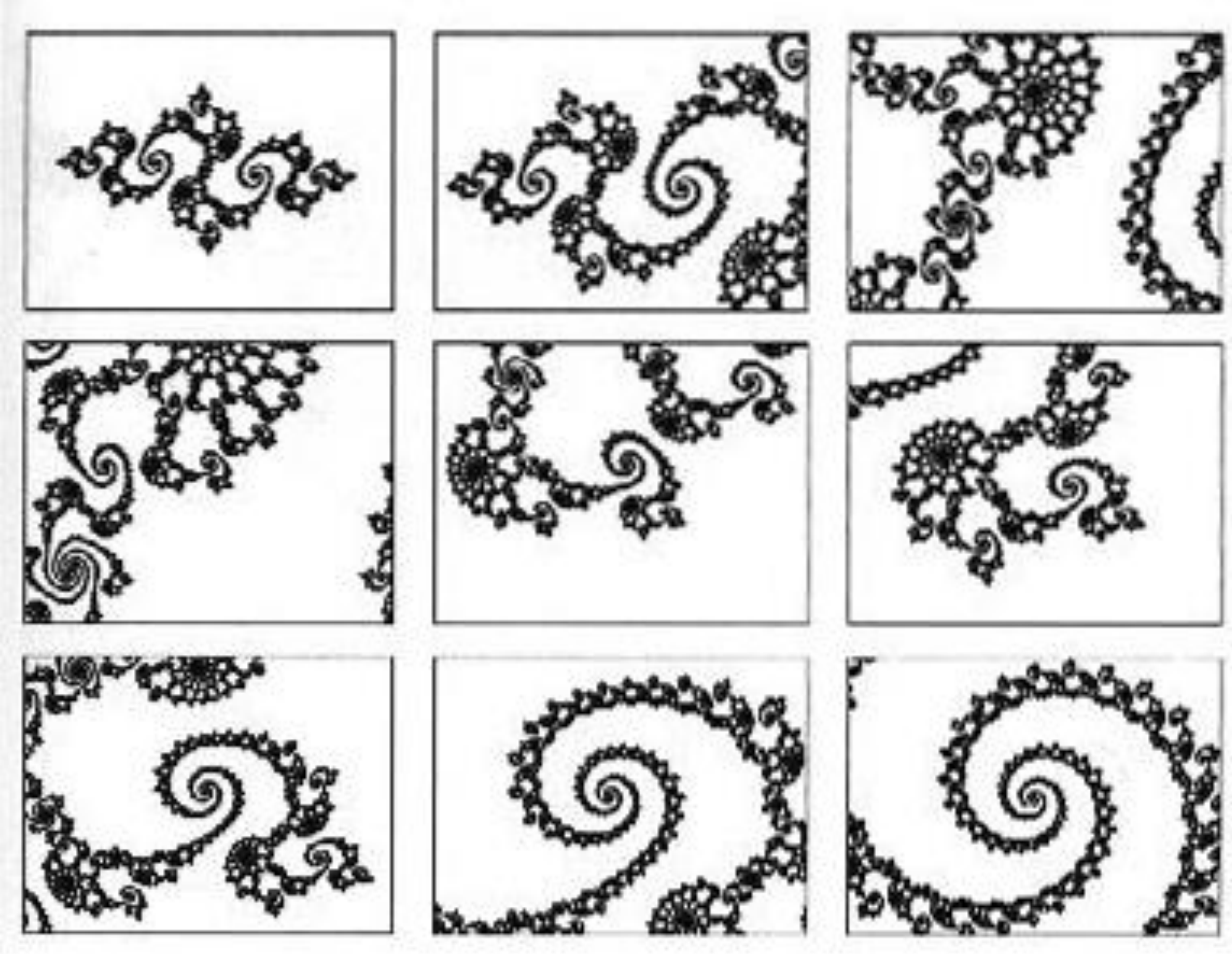
- Generally explicit formulas of these encirclements cannot be given

Encirclement 2





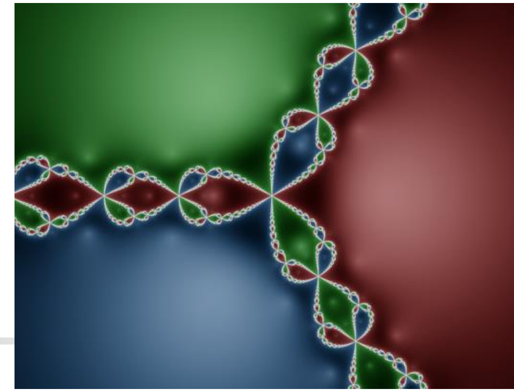
Zooming Julia sets



Connectivity

- Julia set is a nowhere dense set
- Uncountable set (of the same cardinality as the real numbers)
- Can be connected and unconnected (Fatou dust)
- Based on critical orbit:
  - Critical point – where derivation is 0
  - 0 -> c -> $c^2 + c$ -> $(c^2 + c)^2 + c$ -> …
  - This sequence should be bounded

Definitions



- For arbitrary complex rational function f
- Fatou domains F_i
  - Finite number of open sets
  - f behaves in a regular and equal way on F_i
  - The union of all F_i's is dense in complex plane
  - Each F_i contains at least one critical point of f
- Fatou set F(f) – union of all F_i
- Julia set J(f) – complement of F(f)
- Each of the Fatou domains has the same boundary, which consequently is the Julia set
- J(f) is connected <=> each Fatou component contains at most one critical value.

Properties

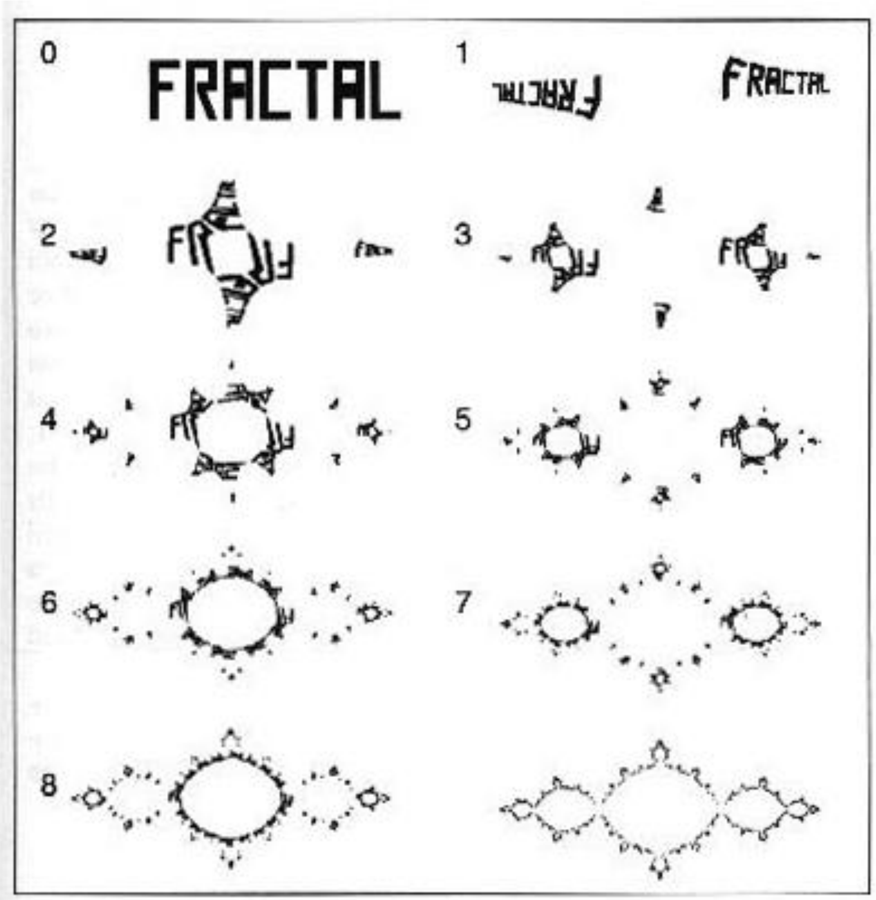
Let J(f) be any Julia set using rational function f. Then:

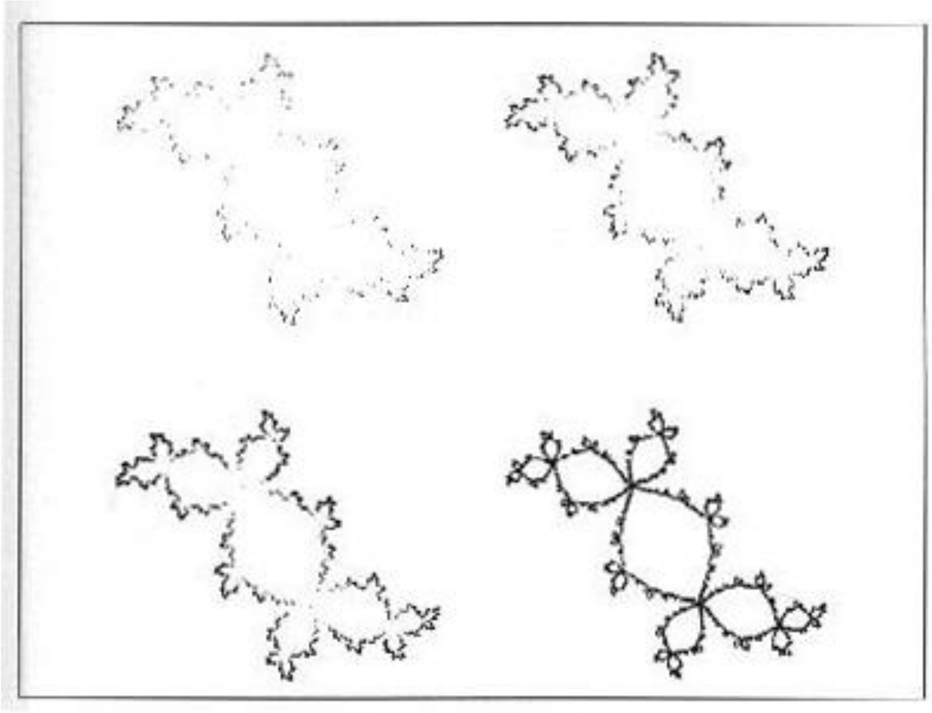
- If a point P belongs to J(f), then all successors (i.e. f(P), f(f(P)), ...) and predecessors of P belong to J(f).
- J(f) is an attractor for the inverse dynamical system of f(z). That means, if you take any point P and calculate its predecessors (predecessors of P are all points Q with f(Q)=P or f(f(Q))=P and so on...), then these predecessors converge to J(f).
- If P belongs to J(f), then the set of all predecessors of P cover J(f) completely.
- If f(z) is a polynomial in z, then J(f) is:
  - either connected (one piece)
  - or a Cantor set (dust of infinitely many points)
- If f(z) is a polynomial in z, then J(f) can be thought of the border of the area defining the set of points which are attracted by infinity.

Drawing similar to IFS

- Using inverse transformations
- 2 functions
- Finding point inside Julia set
- Finding fixed point
- Complete invariance

Using IFS



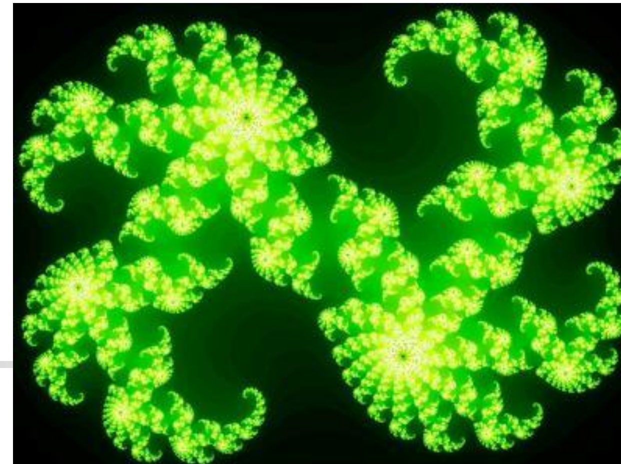


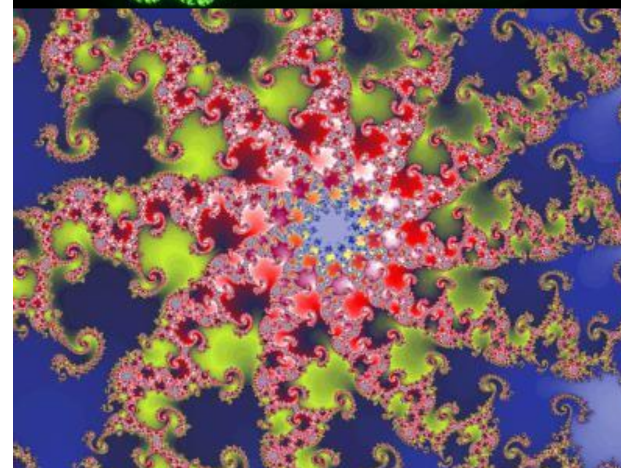
Invariance

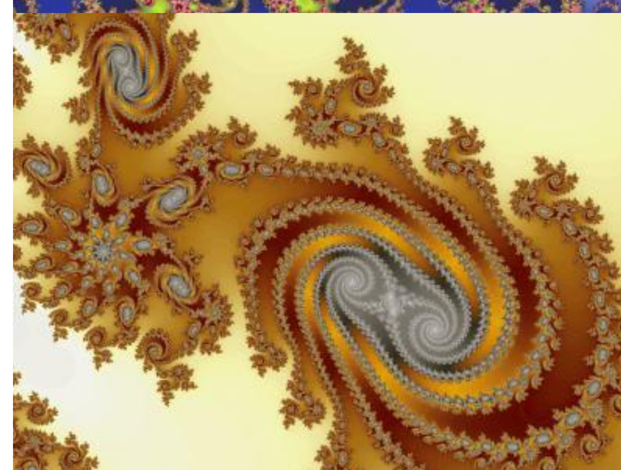
- If z is point from set, also f(z) is from set
- $f^{-1}(z) = -(z-c)^{0,5}$
- $f^{-1}(z) = +(z-c)^{0,5}$
- $f(z) = z^2+c$
- Indicates self-similarity

Visualization

- For each pixel in image, compute iterations, with max number of iterations
- Check for boundary 2
- Color pixel based on total number of iterations for that pixel
- Use color table



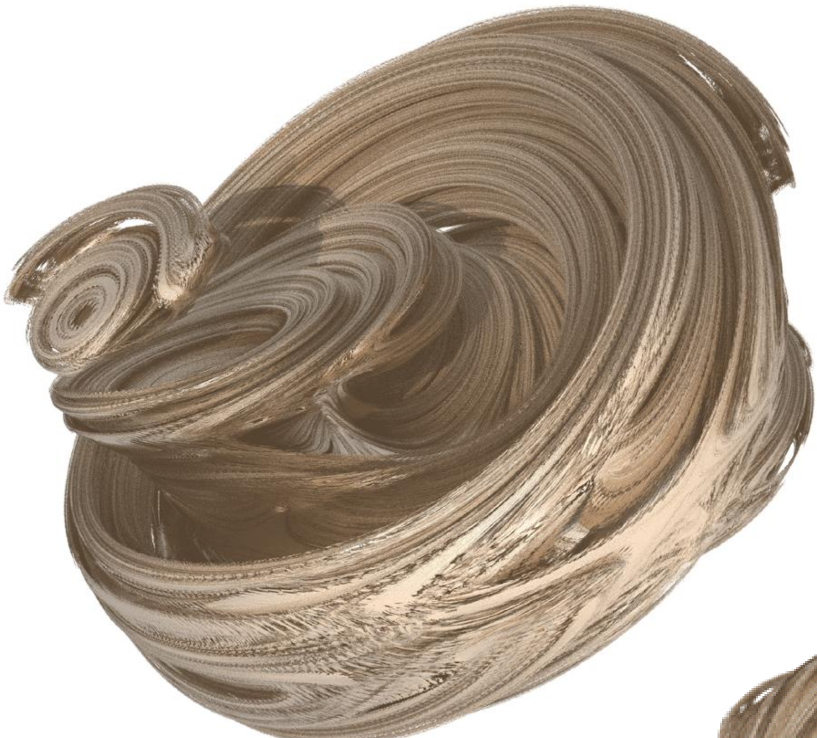




Quaternion Julia sets

- Extension of real and complex numbers
- $i^2=j^2=k^2=ijk=-1$
- $z=x_0+x_1i+x_2j+x_3k$
- Four dimensions
- We can ignore some coordinates
- Again $z \to z^2 + c$

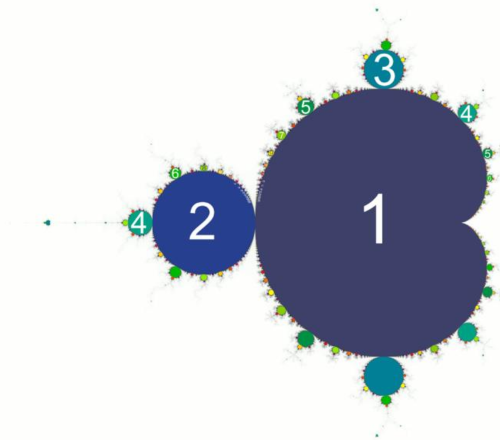
Quaternion Julia sets



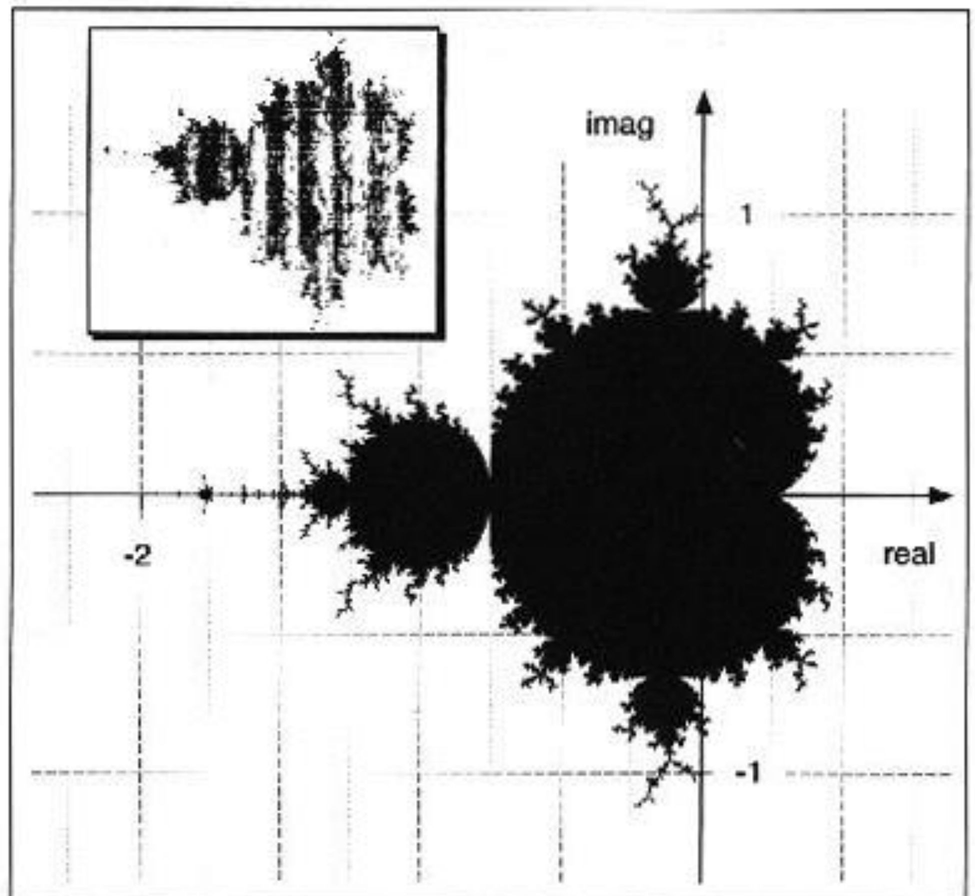
Mandelbrot set

- M={c in C;J_c is connected}
- M={c in C;c->c^2+c->… is bounded}
- Threshold radius 2
- Encirclements
- Not same iterations, these are different for each point

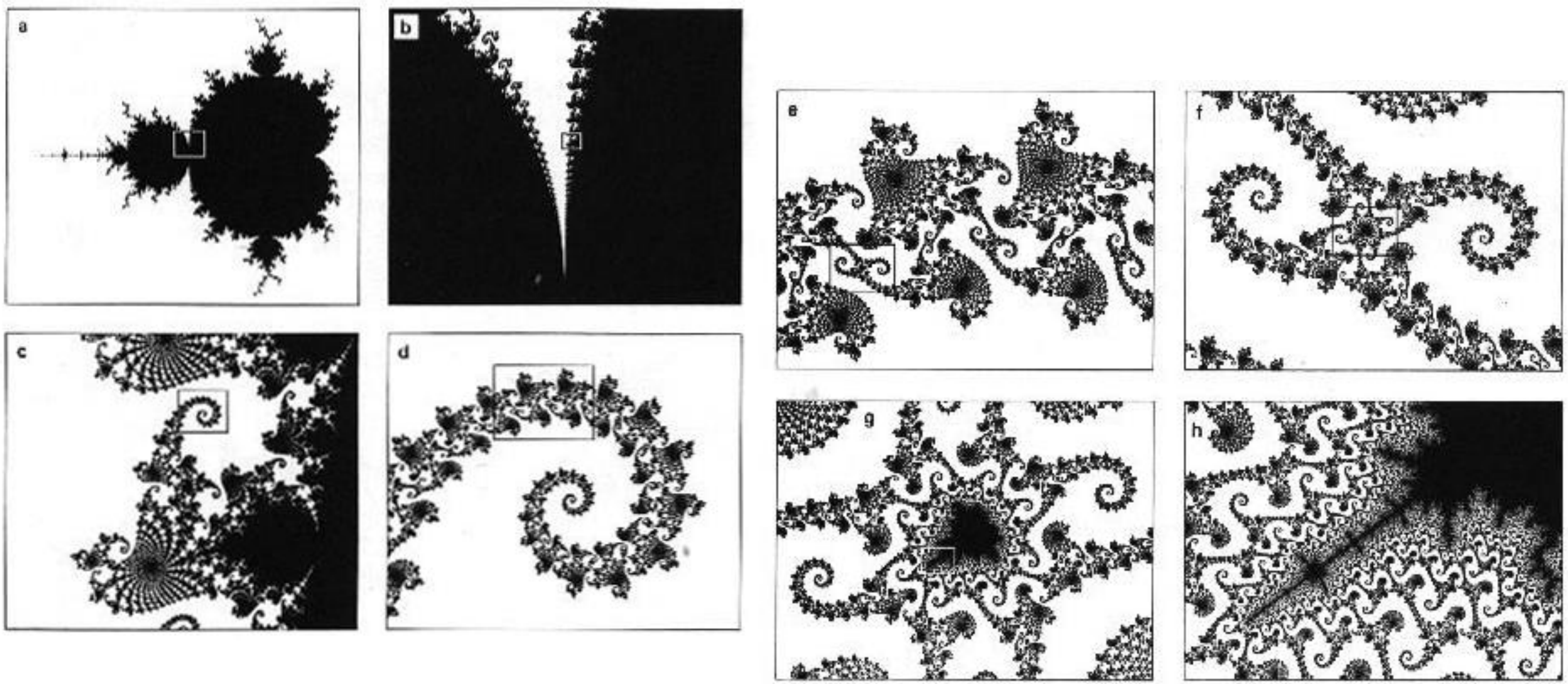
Mandelbrot set 2



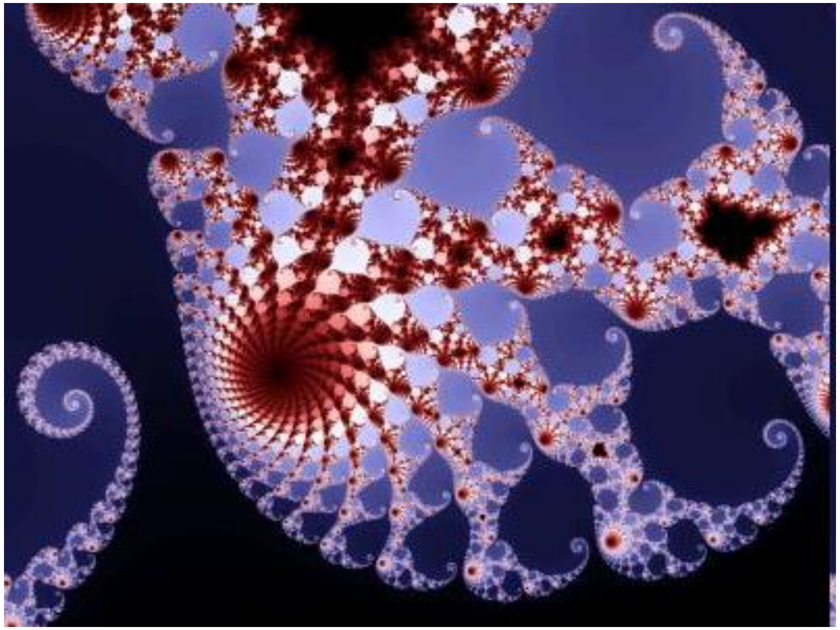
- On real axis
  - [-2, 0.25]
- Area
  - 1.50659177 ± 0.00000008
- Connected
- Haussdorf dimension of boundary – 2
- Period bulbs based on rational numbers



Zooming Mandelbrot



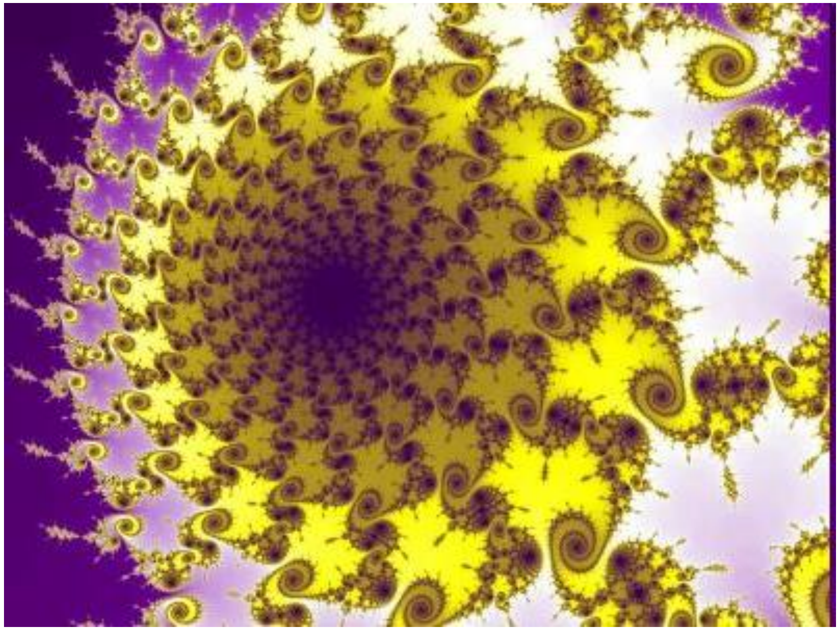
Parts of Mandelbrot



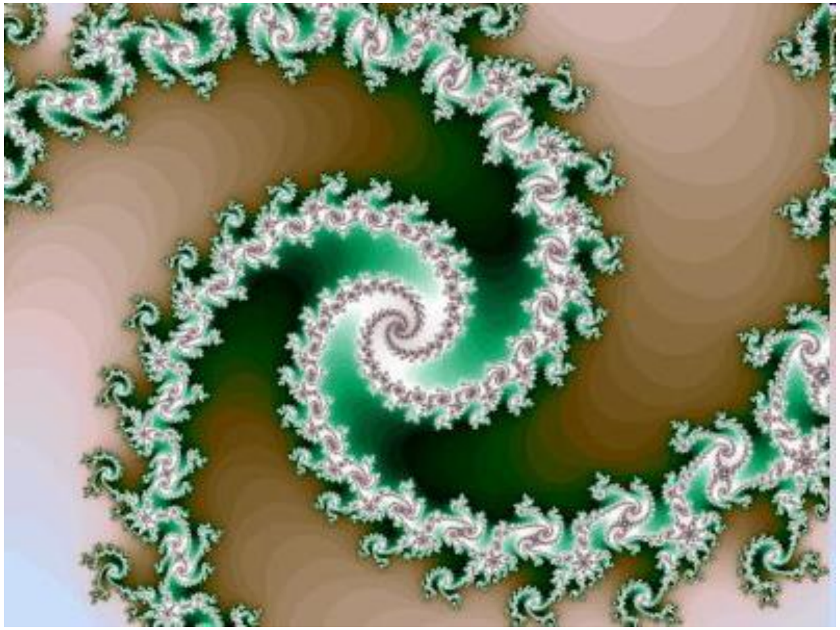
Elephant valley
0,25+0,0i

Seahorse valley
-0,75+0,0i

Parts of Mandelbrot



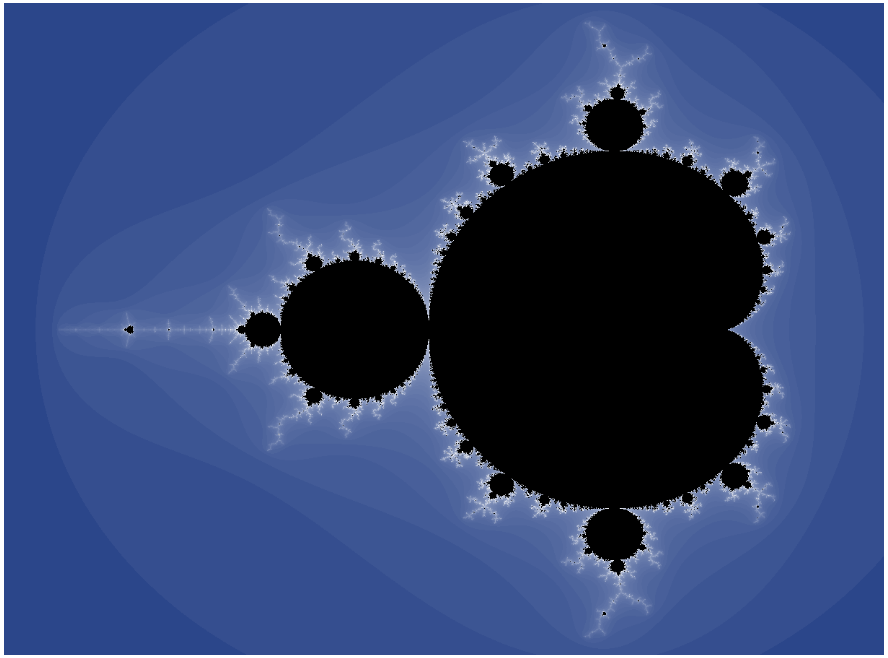
West seahorse valley
-1,26+0,0i

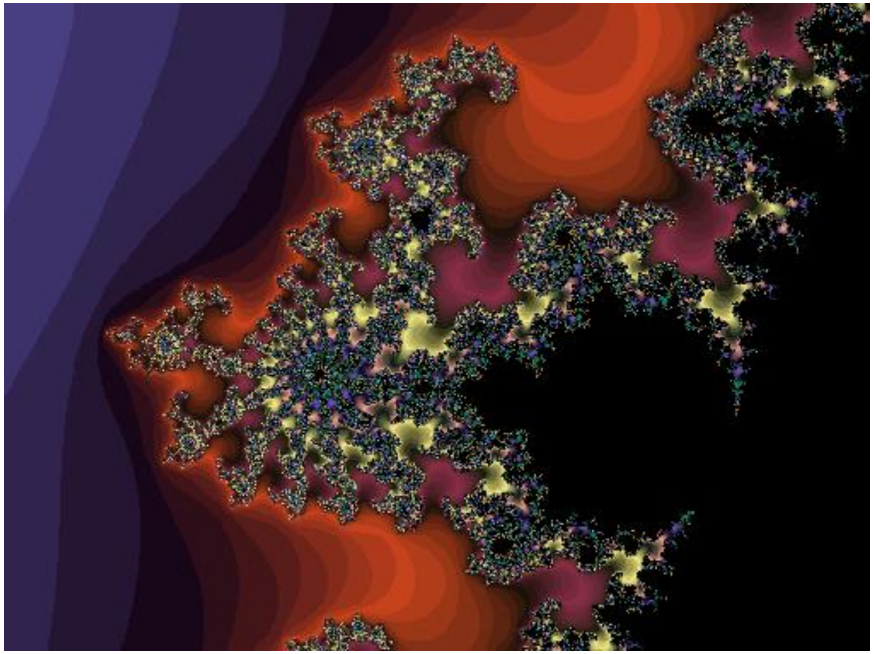


Triple spiral valley
-0,088+0,655i

Coloring Mandelbrot

- Based on number of iterations

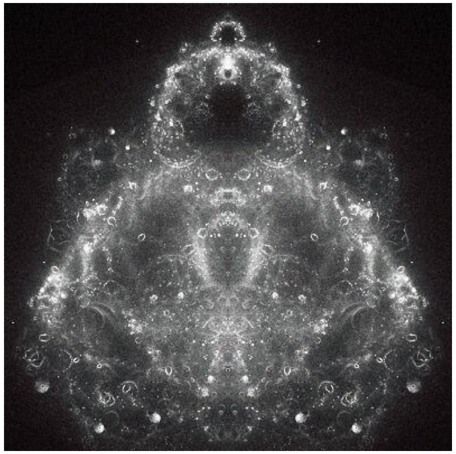


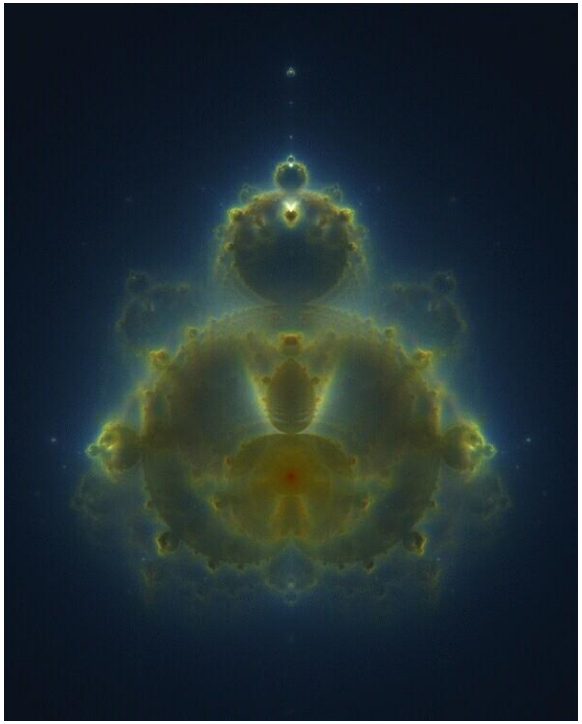


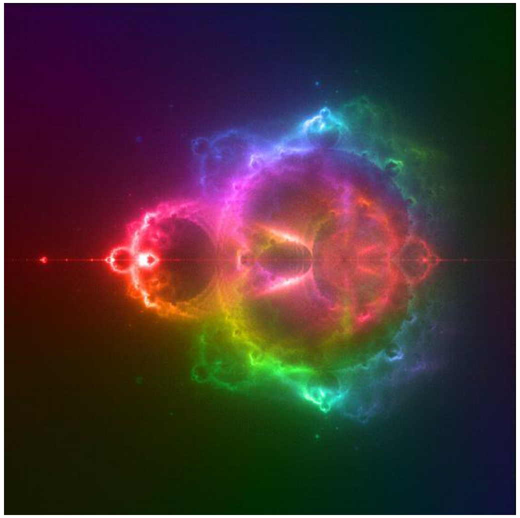
Coloring Mandelbrot

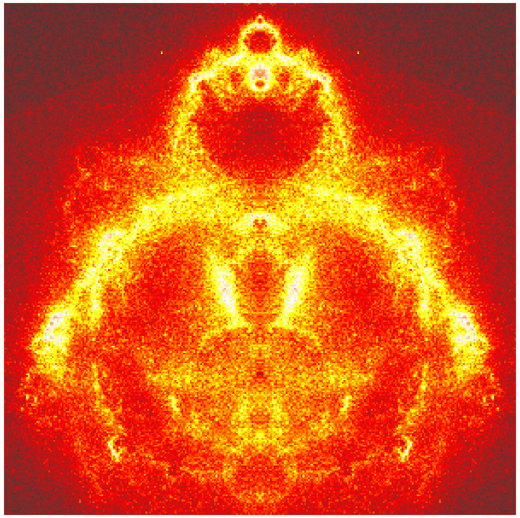
- Coloring by number of iterations
- Coloring by real part of orbit
- Coloring by imaginary part of orbit
- Coloring by sum of real and imaginary part of orbit
- Coloring by angle of orbit

Budhabrot technique

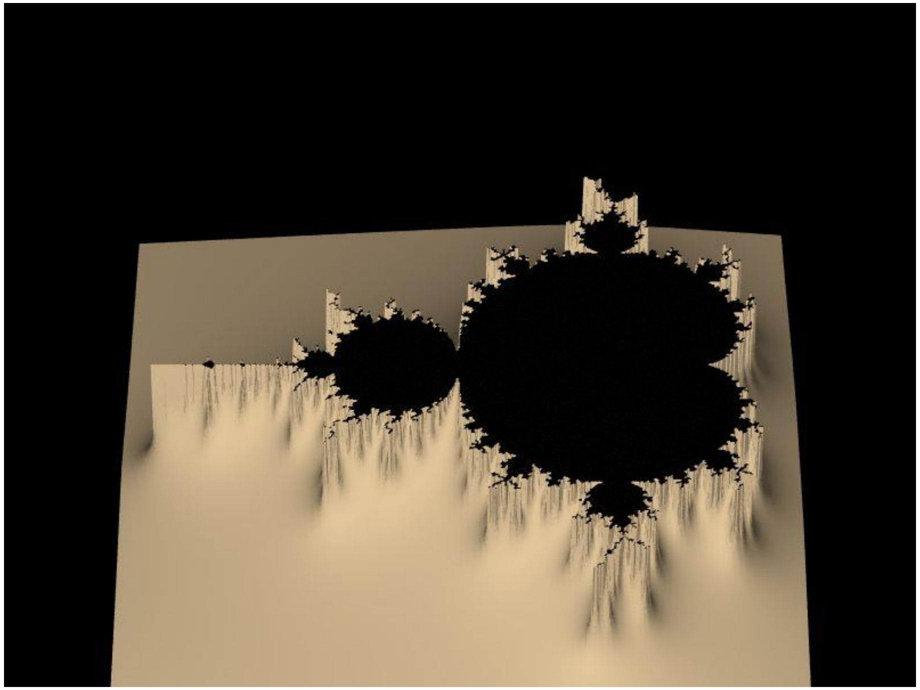


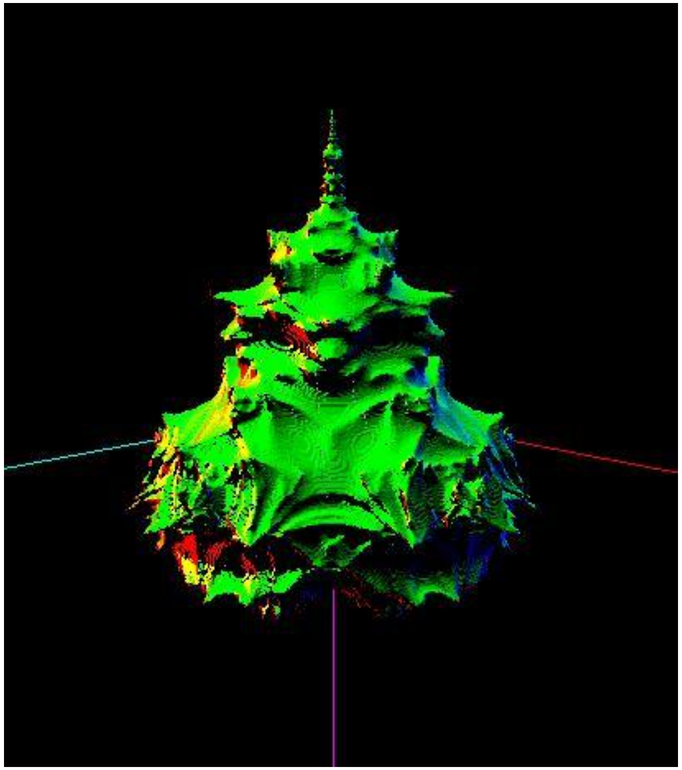




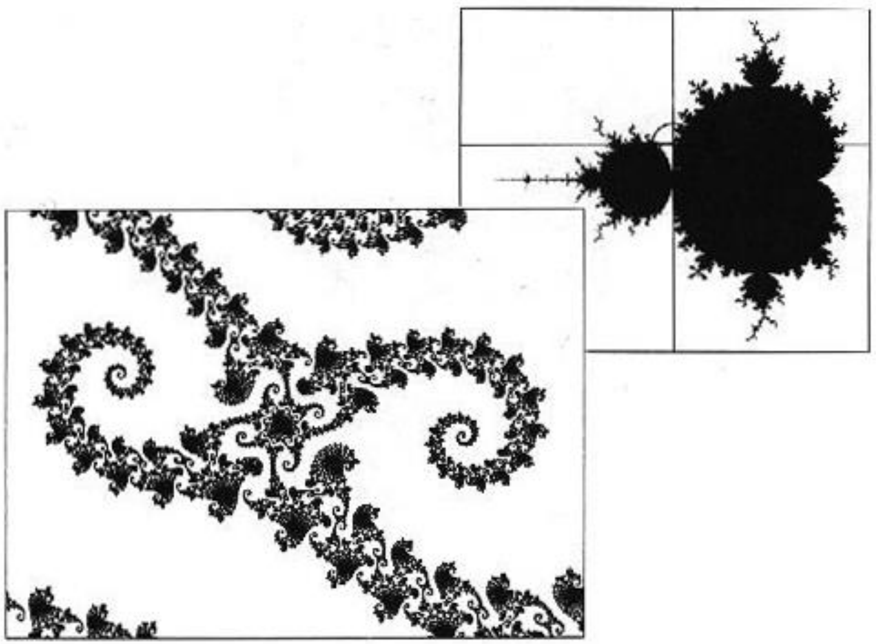


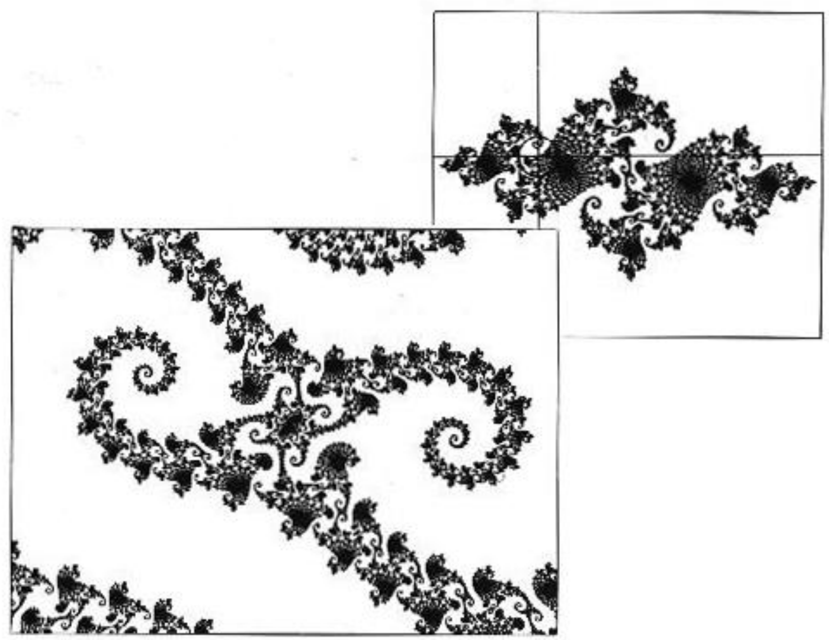
3D Mandelbrot



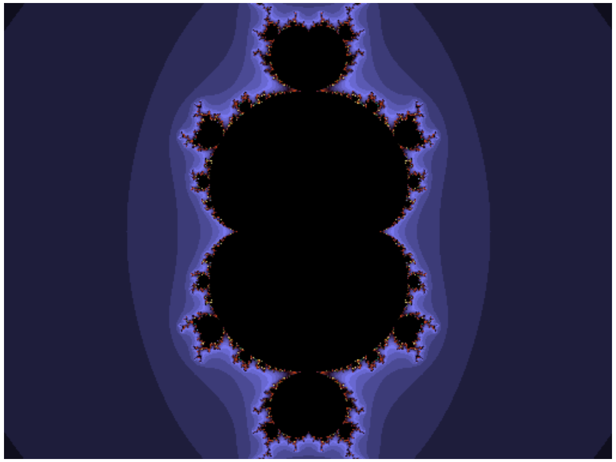


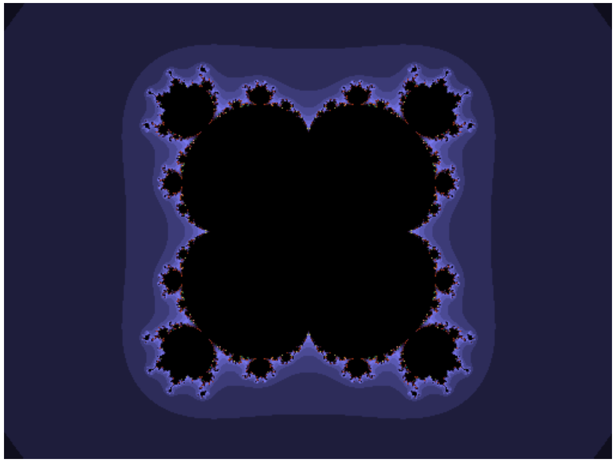
Comparing

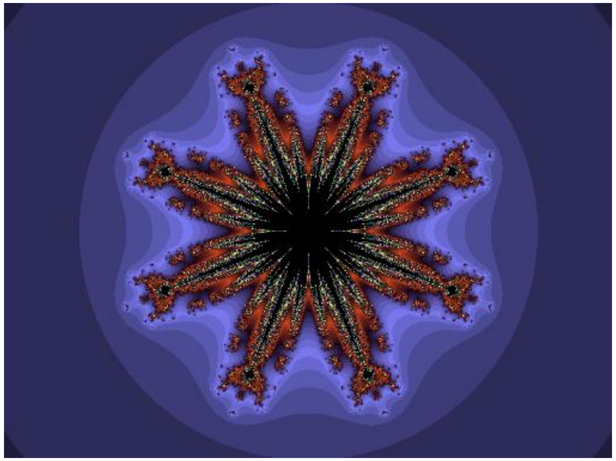


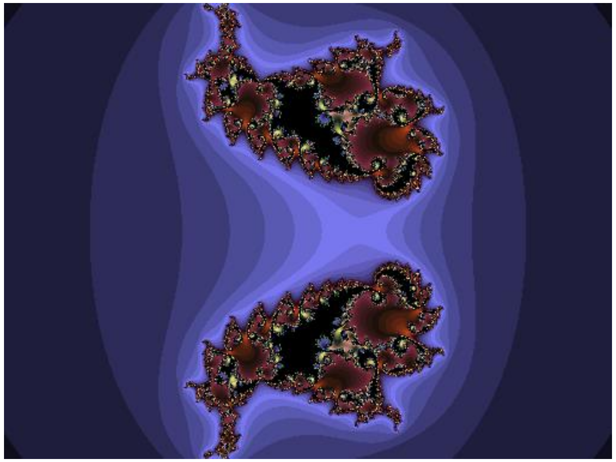


Other formulae

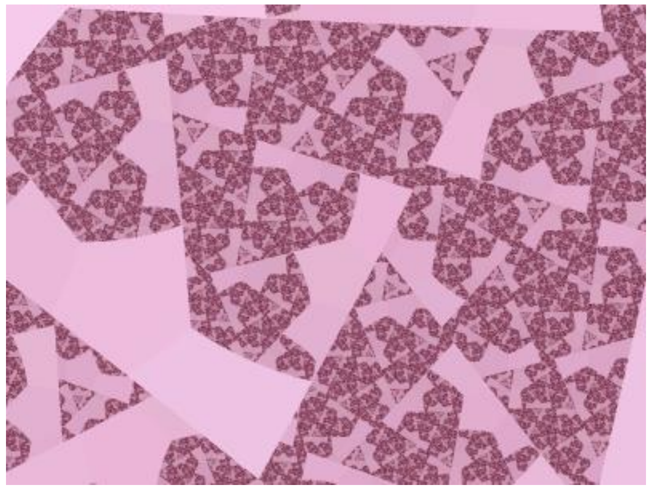


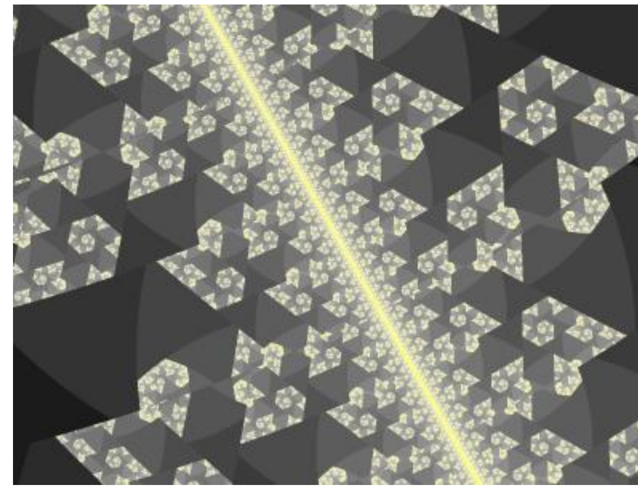


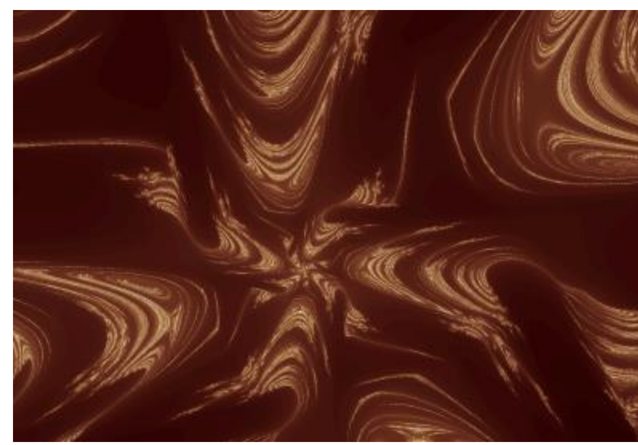




Barnsley M1,M2,M3





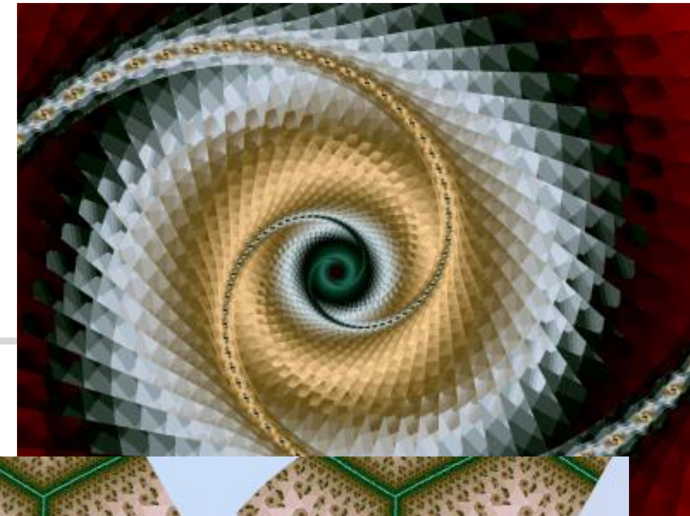


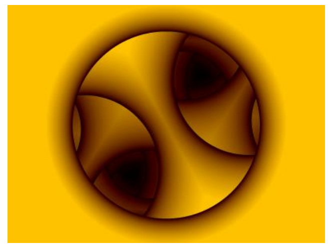
```
if (real(z) >= 0)
        z(n+1)=(z-1)*c
else
        z(n+1)=(z+1)*c
```

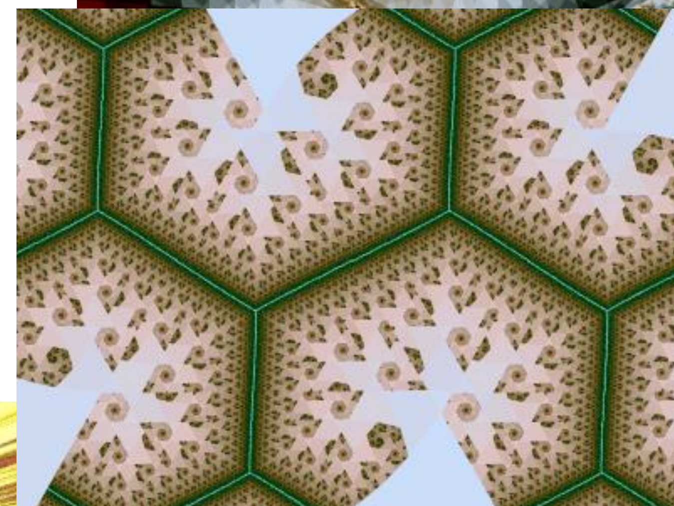
```
if (real(z)*imag(c) + real(c)*imag(z) >= 0)
        z(n+1) = (z-1)*c
else
        z(n+1) = (z+1)*c
```

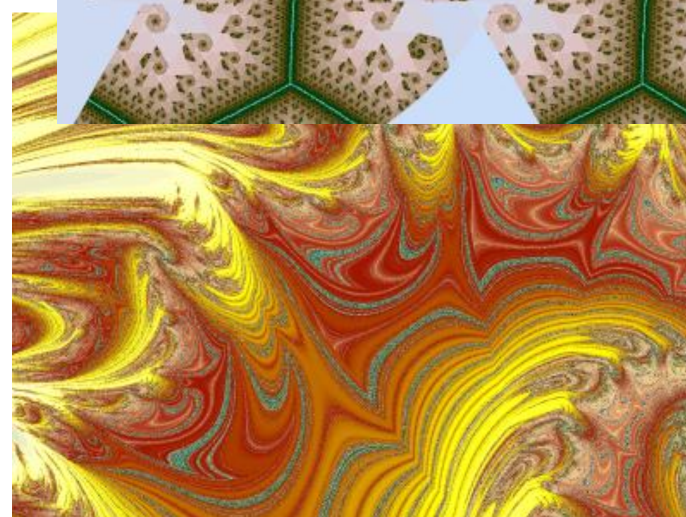
```
if (real(z(n) > 0)
        z(n+1) = (real(z(n))^2 - imag(z(n))^2 - 1) + i * (2*real(z((n)) * imag(z((n)))
else
        z(n+1) = (real(z(n))^2 - imag(z(n))^2 - 1 + real(c) * real(z(n)) + i * (2*real(z((n)) * imag(z((n)) + imag(c) * real(z(n))
```

Barnsley J1,J2,J3









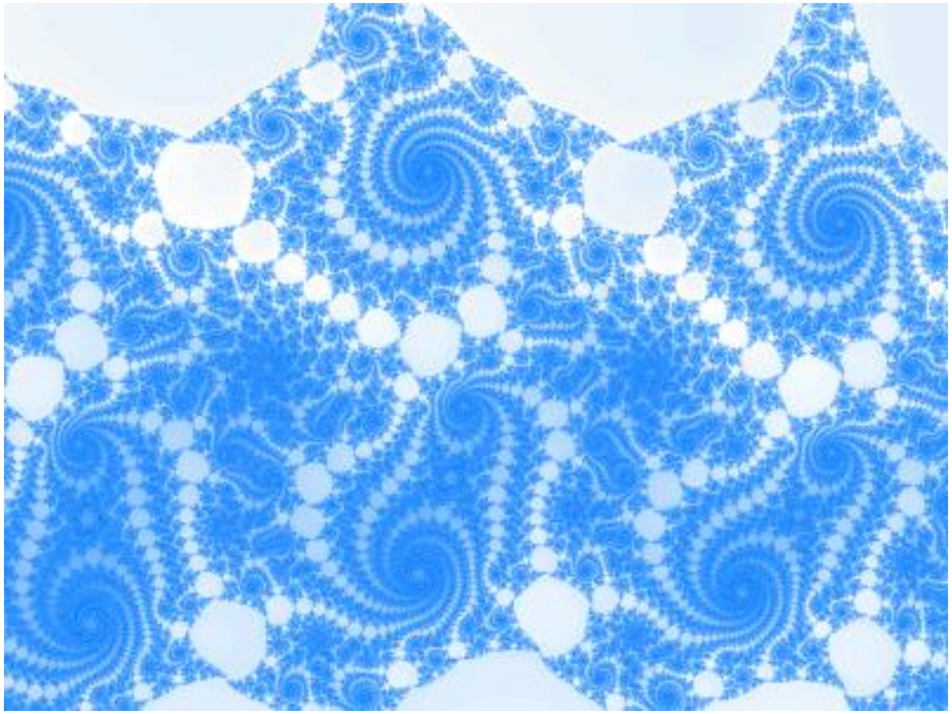
- if (real(z) >= 0)
- z(n+1)=(z-1)*c
- else
- z(n+1)=(z+1)*c
- if (|z|>2) break;

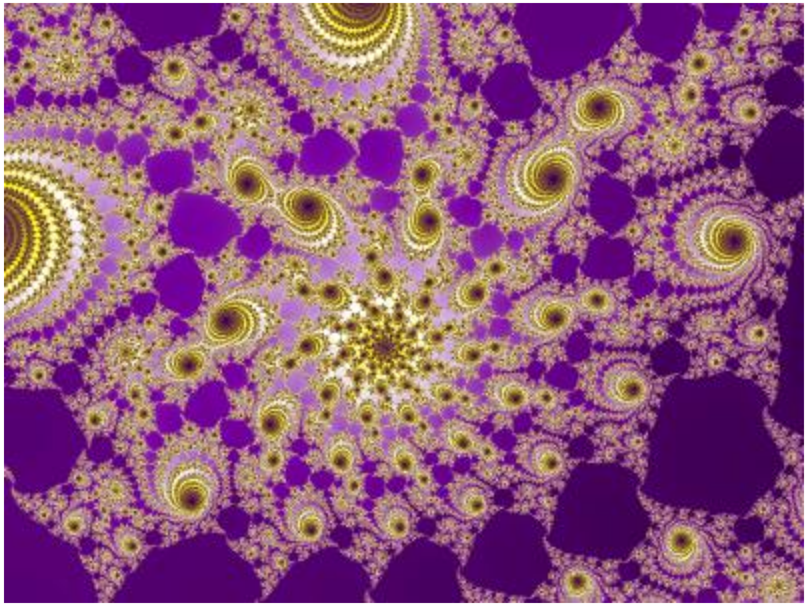
- if (real(z)*imag(c) + real(c)*imag(z) >= 0)
- z(n+1) = (z-1)*c
- else
- z(n+1) = (z+1)*c
- if (|z|>2) break;

- if (real(z(n) > 0)
- z(n+1) = (real(z(n))^2 - imag(z(n))^2 - 1)
- + i * (2*real(z((n)) * imag(z((n)))
- else
- z(n+1) = (real(z(n))^2 - imag(z(n))^2 - 1 + real(c) * real(z(n))
- + i * (2*real(z((n)) * imag(z((n)) + imag(c) * real(z(n))
- if (|z|>2) break;

Magnet

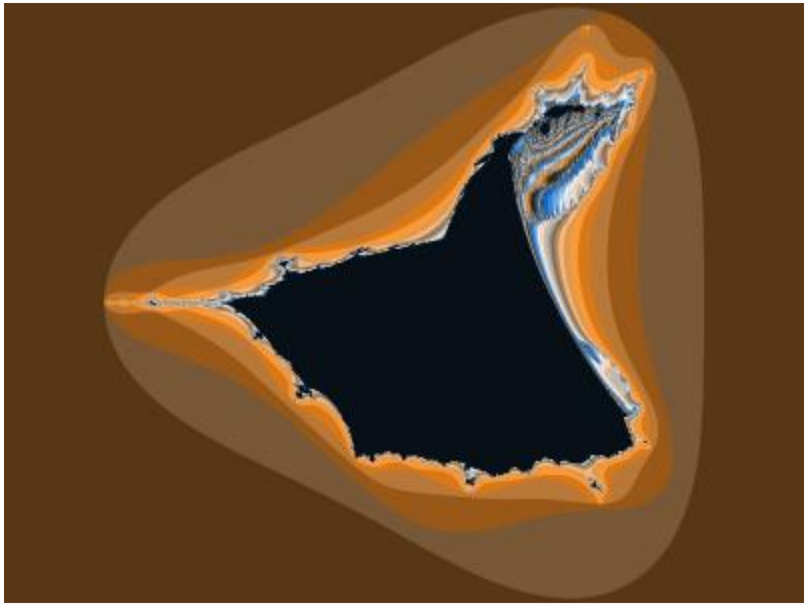
$$z_{n+1} = \left(\frac{z_n^2 + (c-1)}{2z_n + (c-2)}\right)^2$$





Phoenix

$z_{n+1}=z_n^2+Re(c)+Im(c)y_n$
$y_{n+1}=z_n$



End

End of Part 6