Fractals

Part 8 : Stochastic fractals

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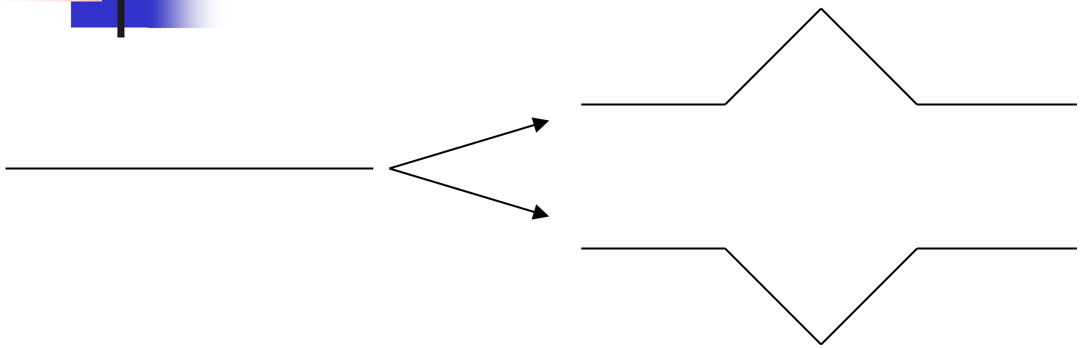
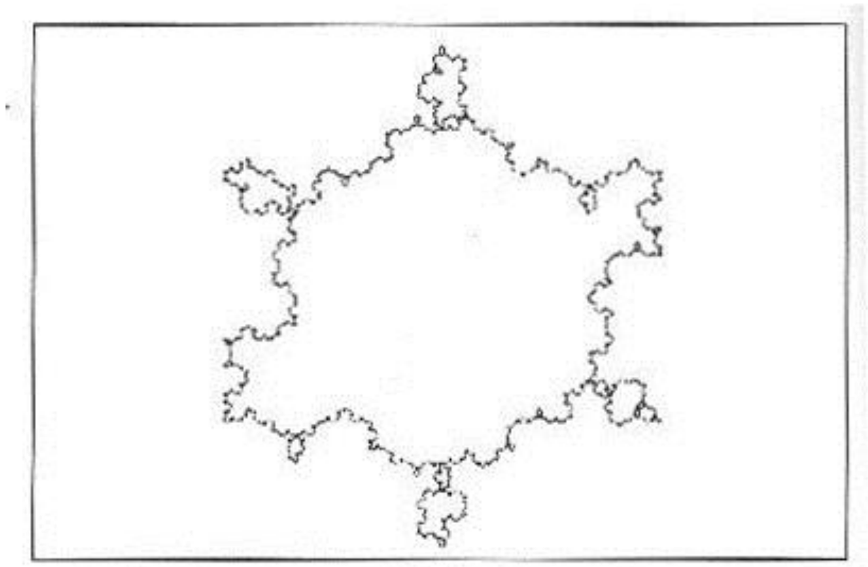
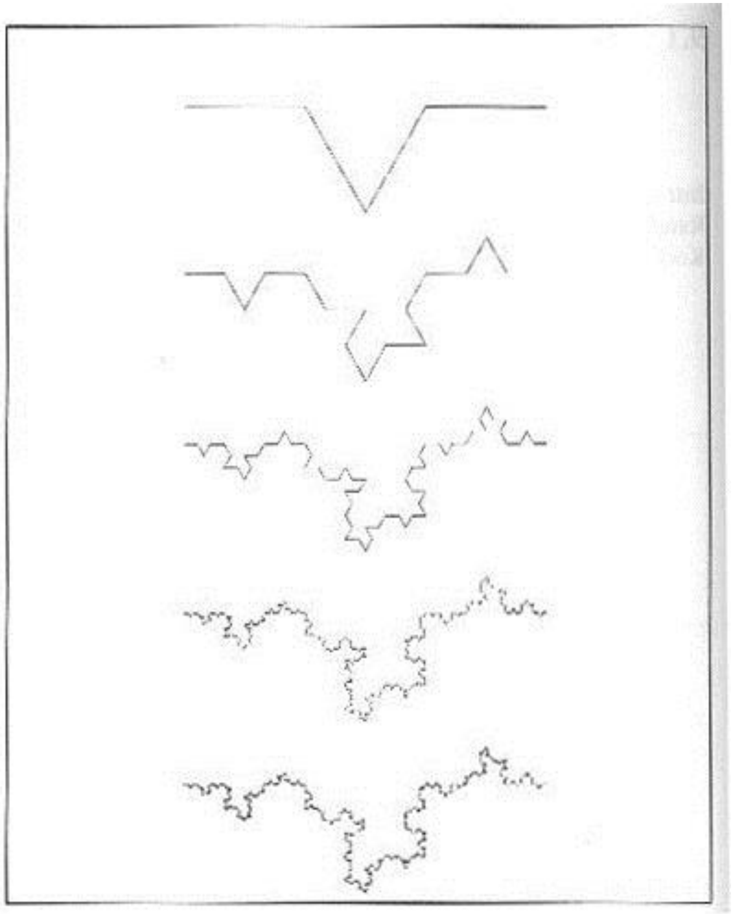
Why randomness ?

- Generalized set of shapes
- More nature-like
- Not strict self-similarity
- Often using Brownian motion
- For each type of fractal
- Using randomness in any stage

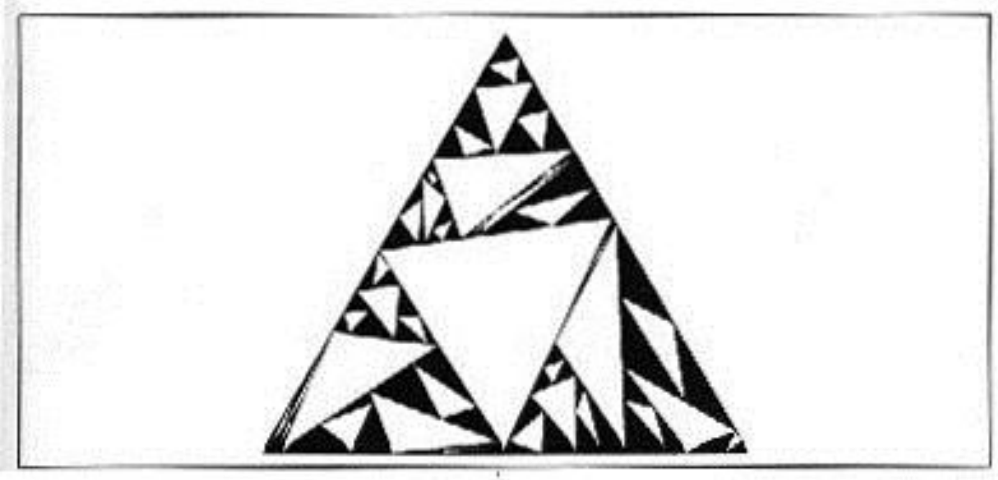
Classical fractals - Koch

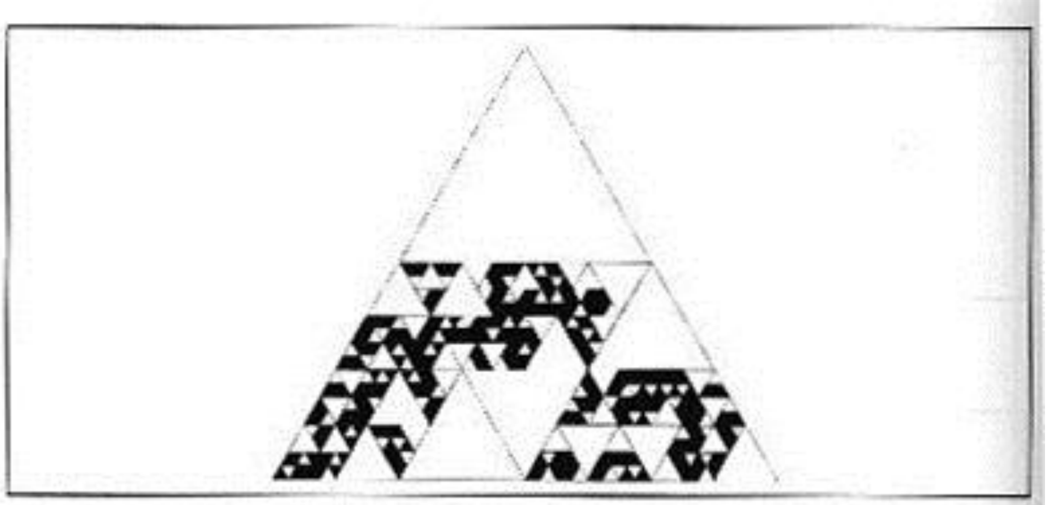






Sierpinski

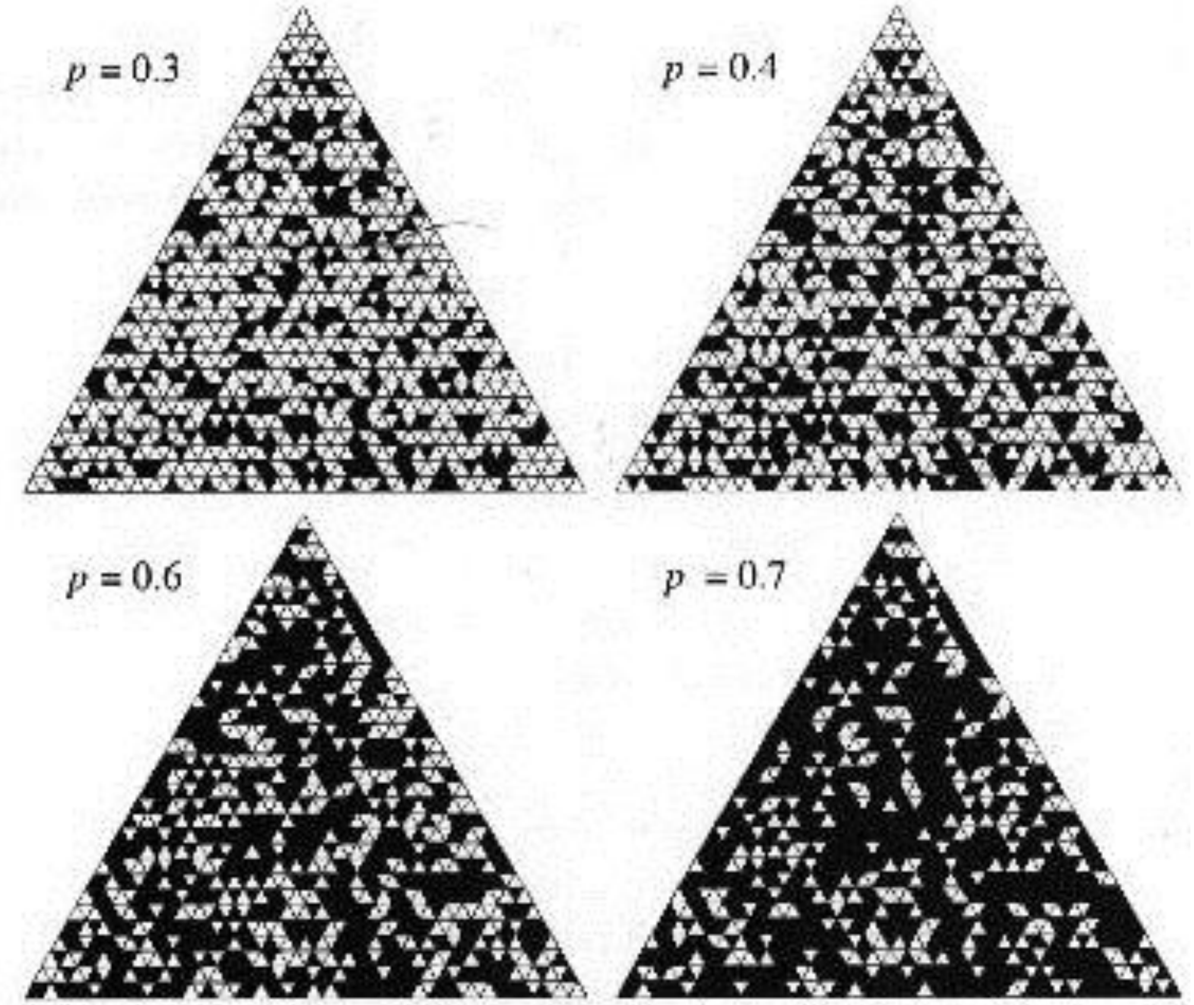




Percolation

- Given triangular or square lattice
- Given probability *p*
- Color each sub cell with probability
- Check number of disjunctive parts
- Many -> one clusters = percolation
- p_c – percolation threshold

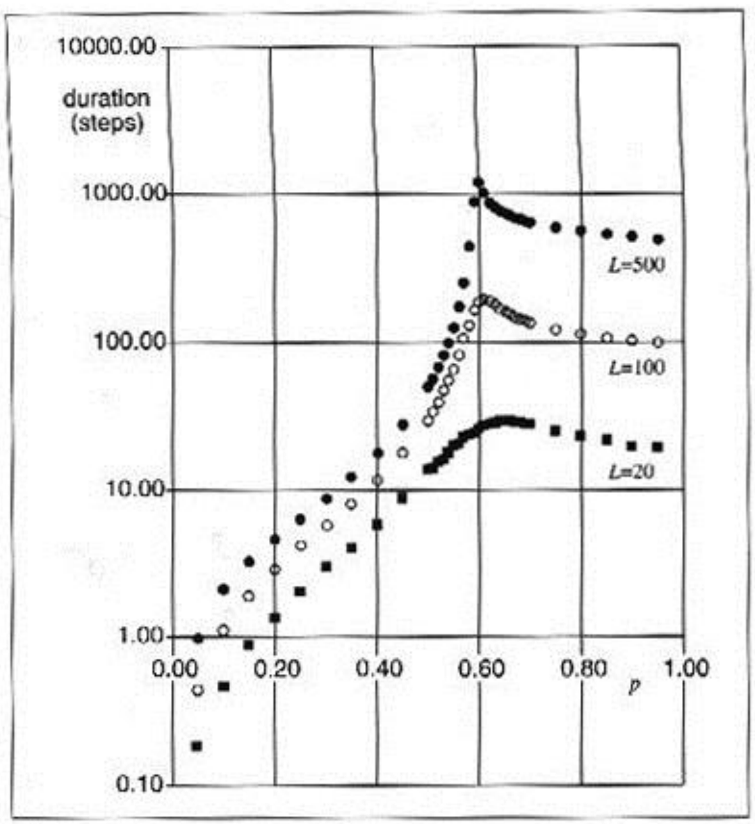
Percolation 2

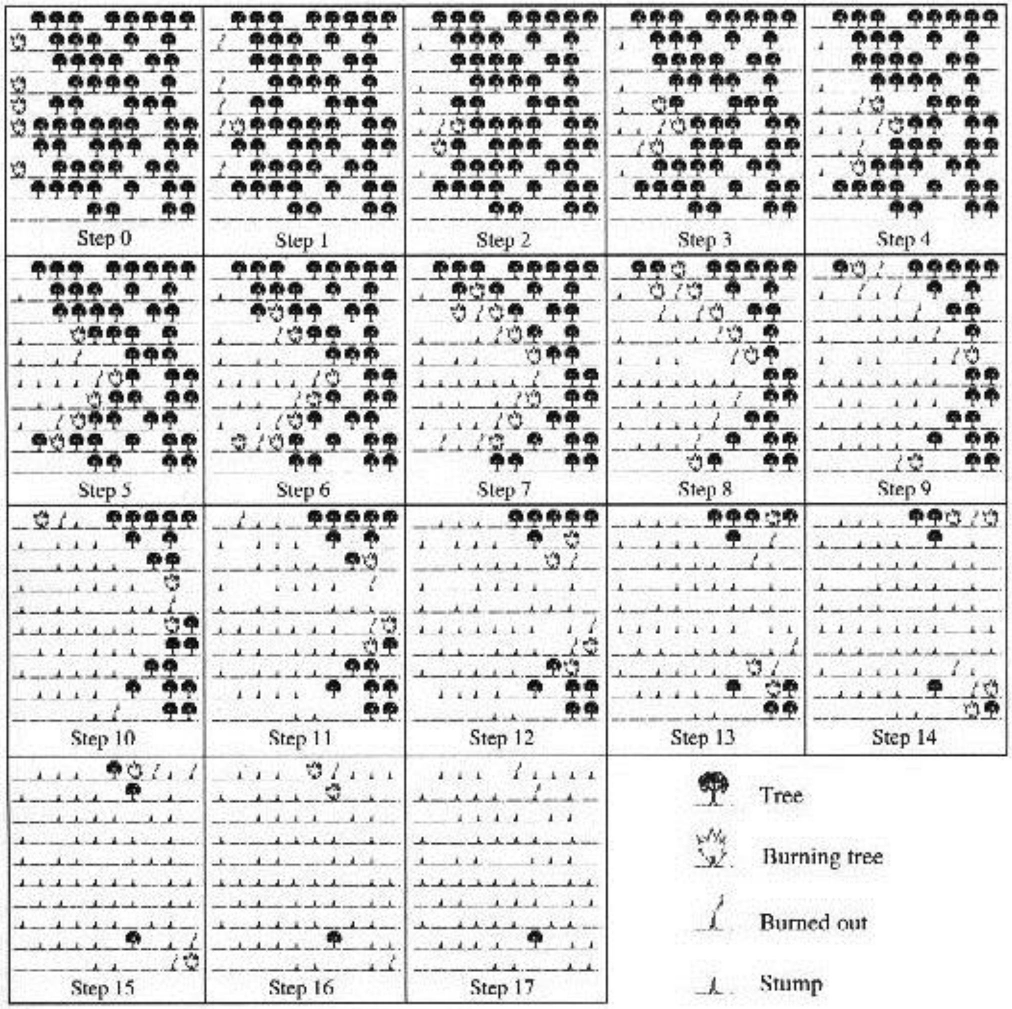


Forest fire simulation

- Square lattice
- Trees with probability p
- For $p>p_c$ whole forest will burn
- For $p<p_c$ only part of forest will burn
- For $p=p_c$ the forest will burn for longest time
- $p_c \sim 0.5928$

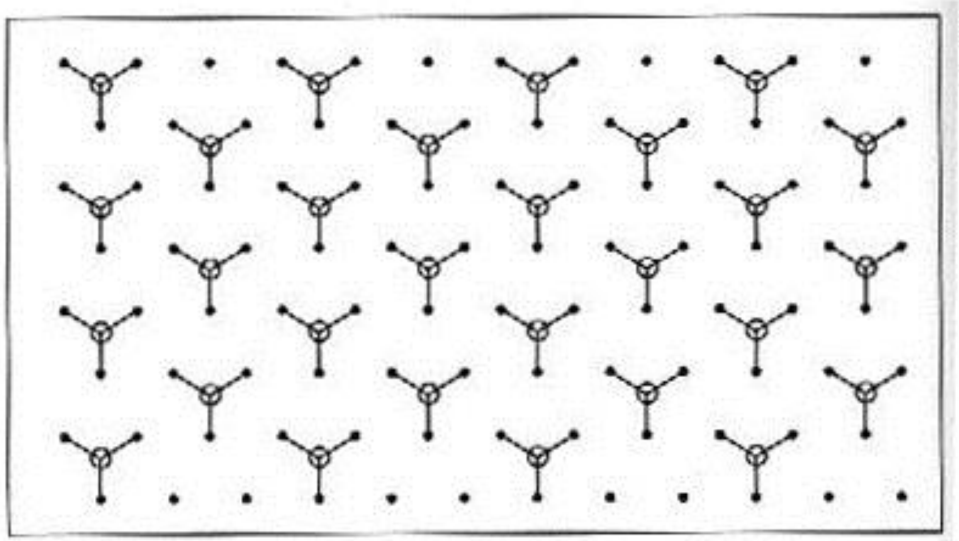
Forest fire simulation 2





Renormalization

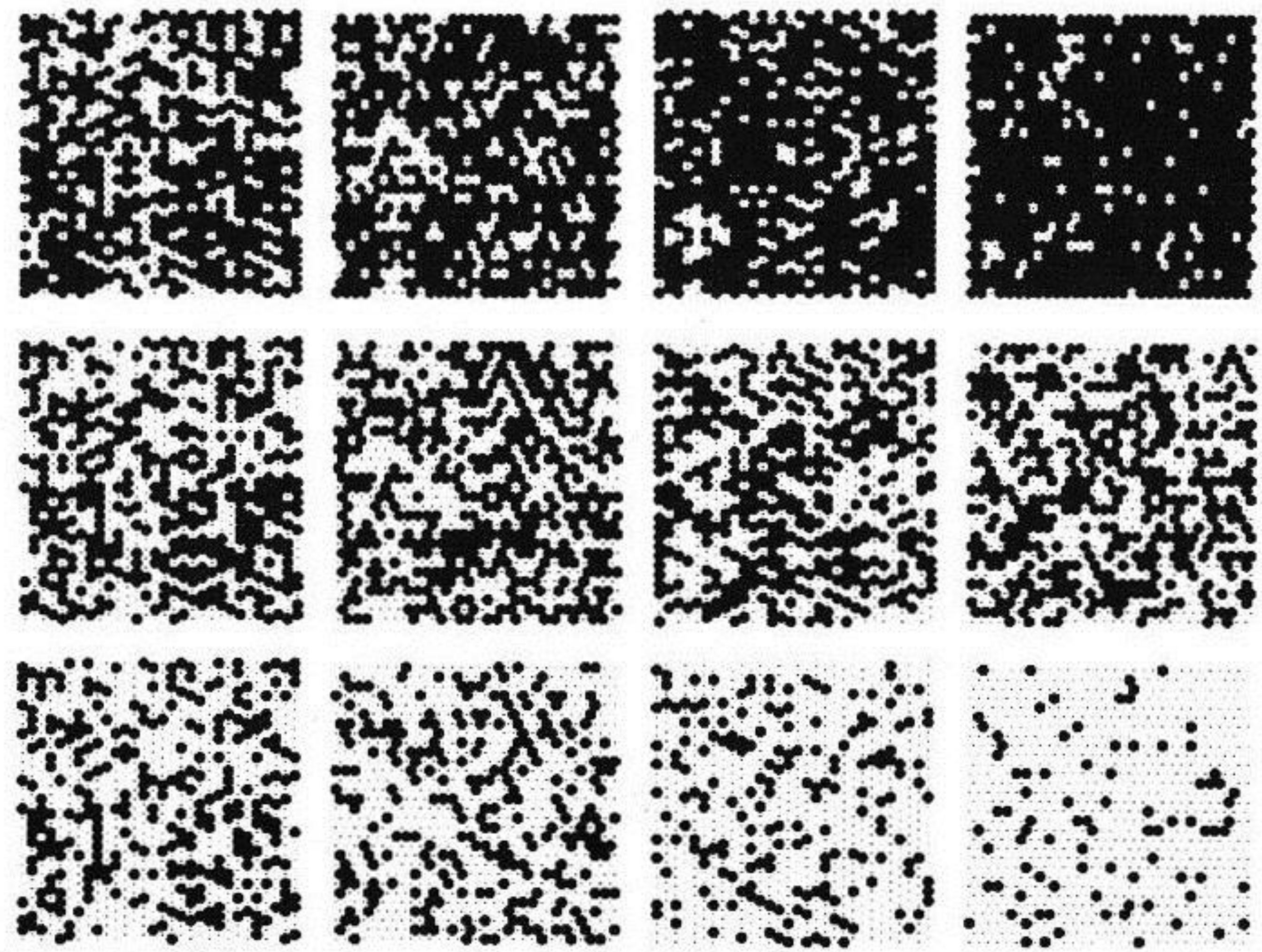
- For triangular lattice
- Sites -> super-sites
- Super-site is occupied if two or three sites are occupied



Renormalization 2

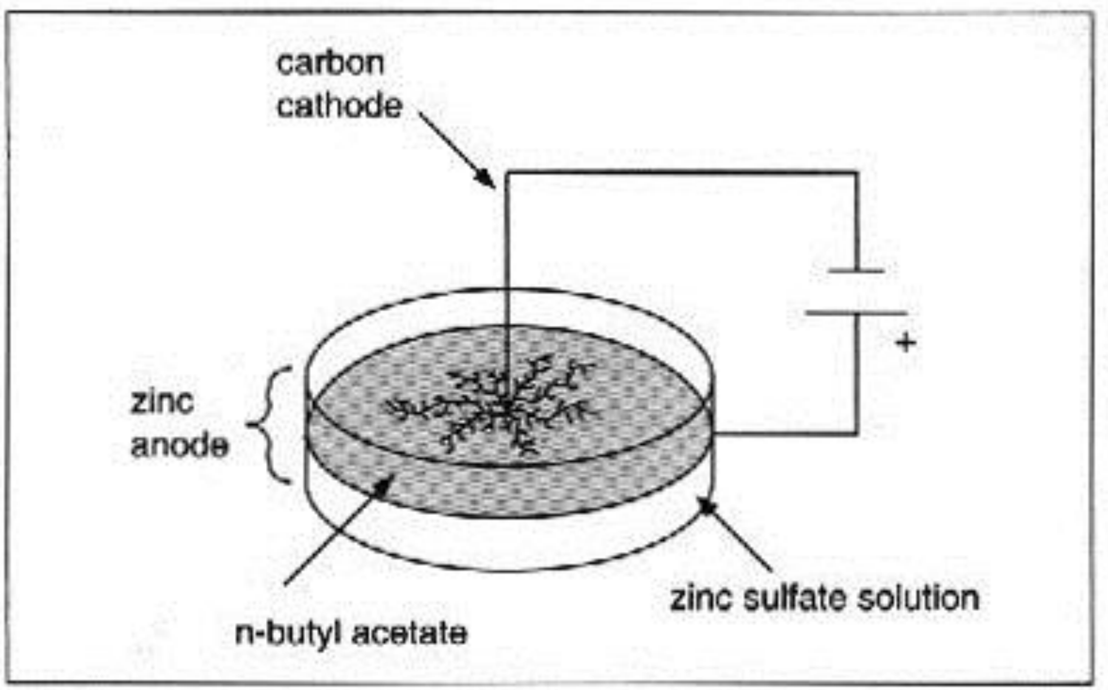
- With p’>p we fill gasps
- With p’<p clusters will vanish
- p’=p we expect similarity
- $p' = p^3 + 3p^2(1-p)$
- $p = p_c = 0.5$
- Statistical self-similarity

Renormalization 3

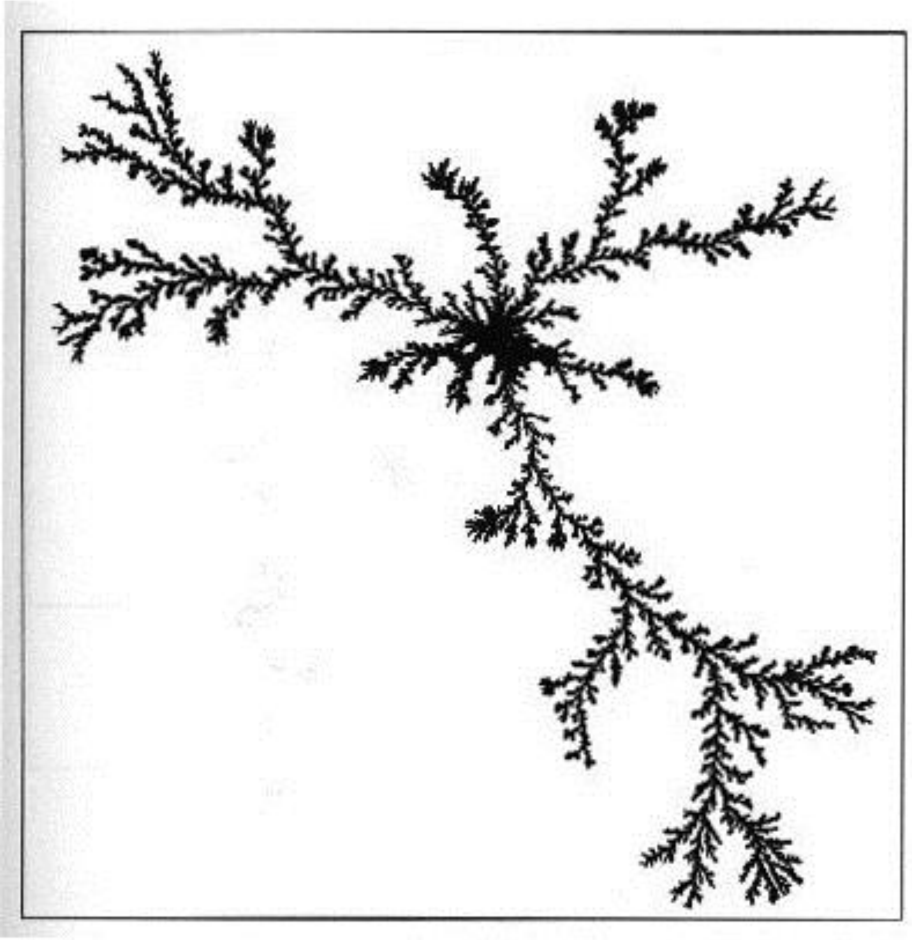


Particles aggregation

- Laboratory experiment
- Zinc-metal leaves



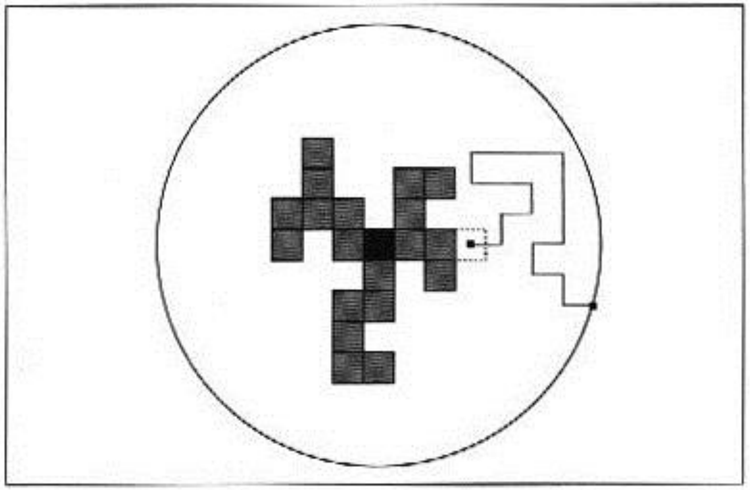
Result

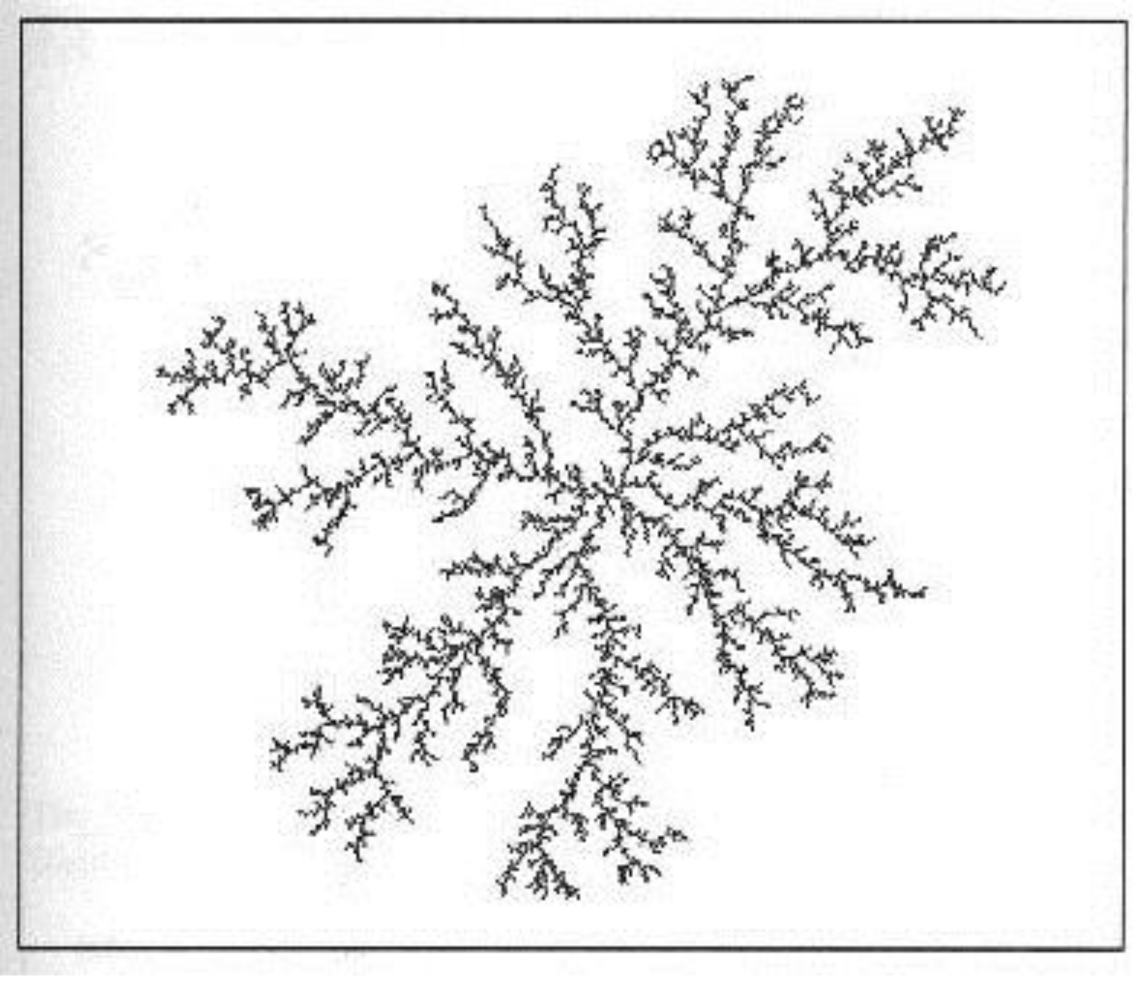


DLA

- Particle is moving with Brownian motion
- If free particle approaches to sticky particle, it stops and becomes sticky
- Repeating with another particle
- Simulation using pixels
- Diffusion Limited Aggregation

Simulating DLA



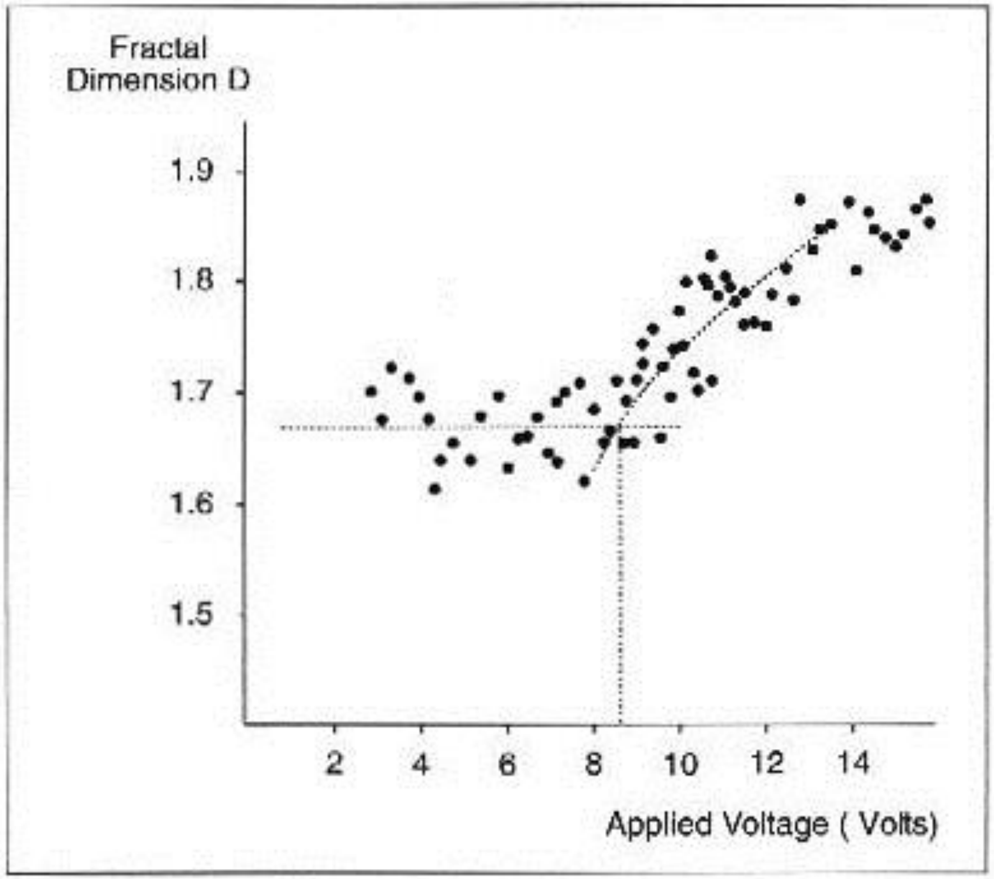


Problems

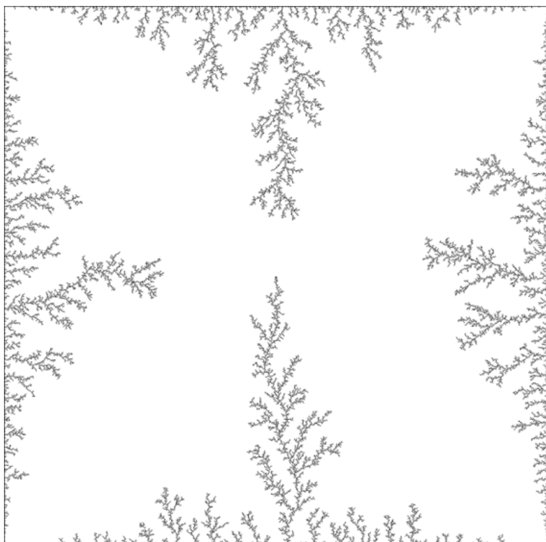
- What is the fractal dimension?
- Density of particles decreases from center. Is there power law for it?
- Is voltage with relation to fractal dimension?
- Is size of aggregate with relation to fractal dimension?

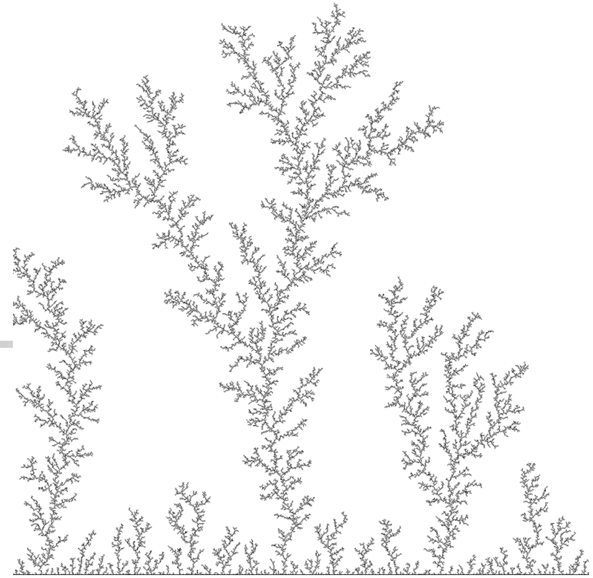
Problems 2

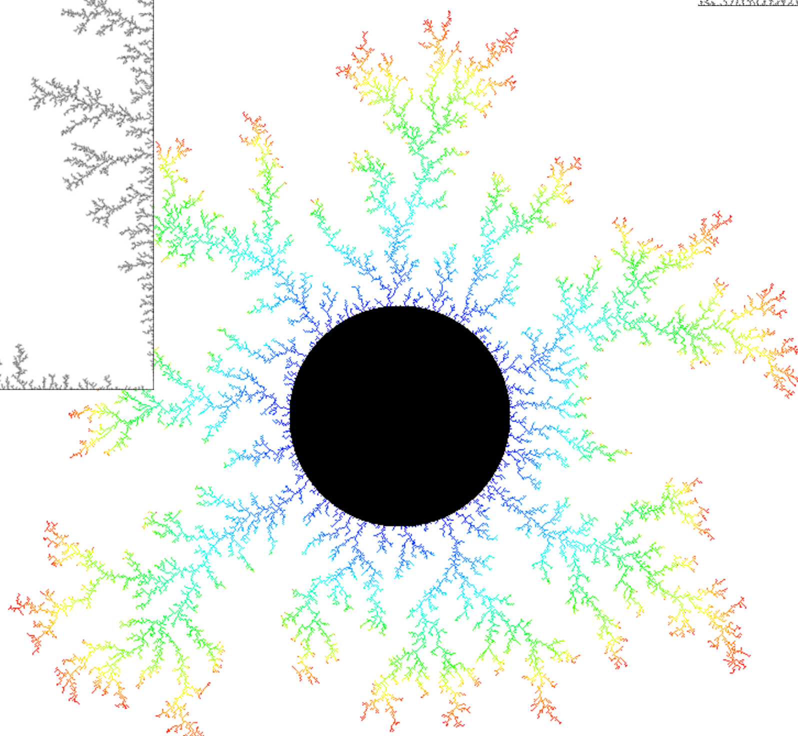
- Still not precise solutions
- D~1.7

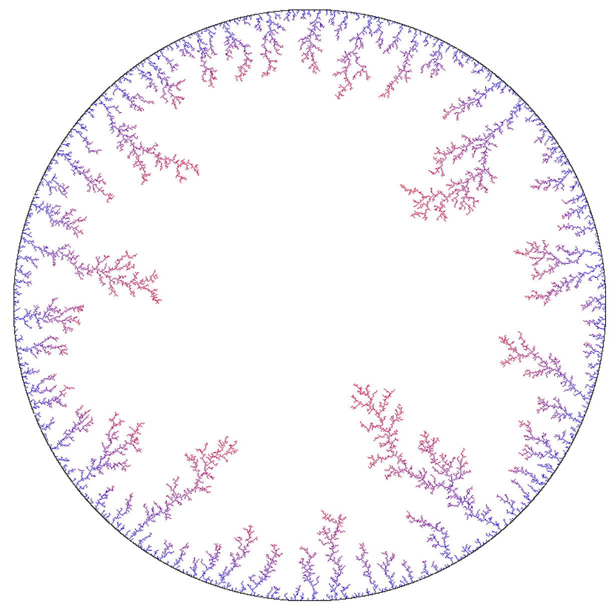


Other DLAs



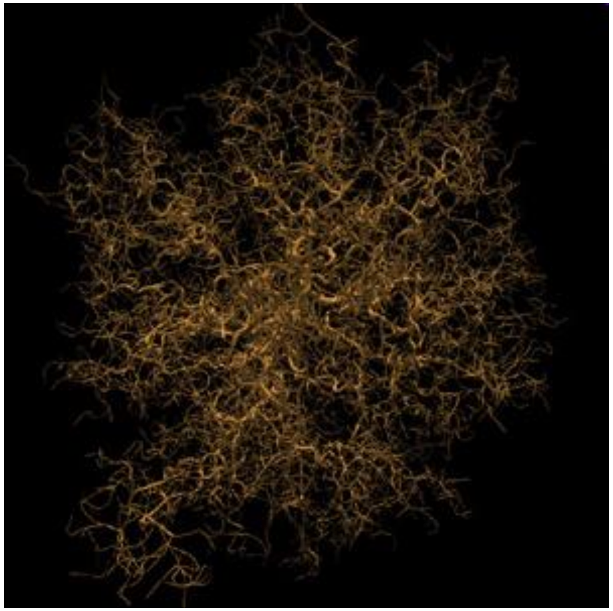


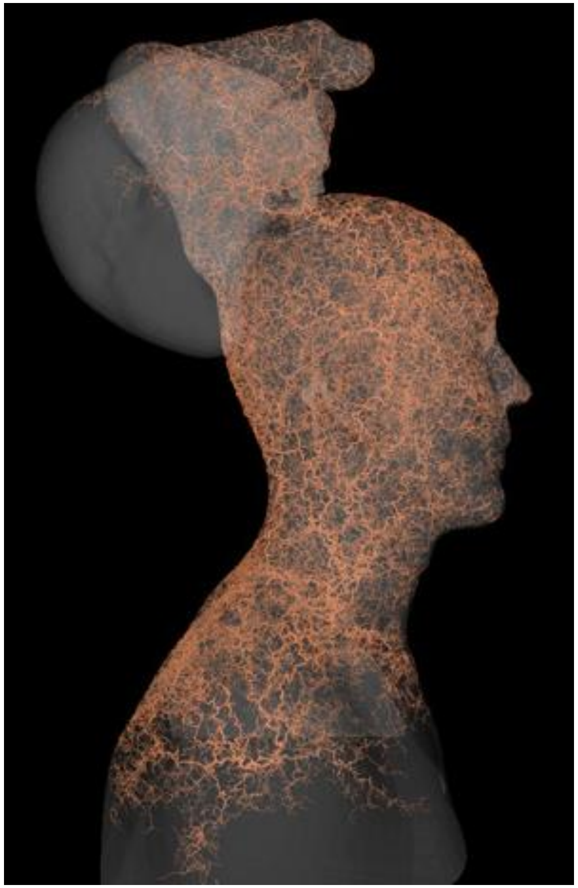




3D DLA

- local.wasp.uwa.edu.au/~pbourke/fractals/dla3d/



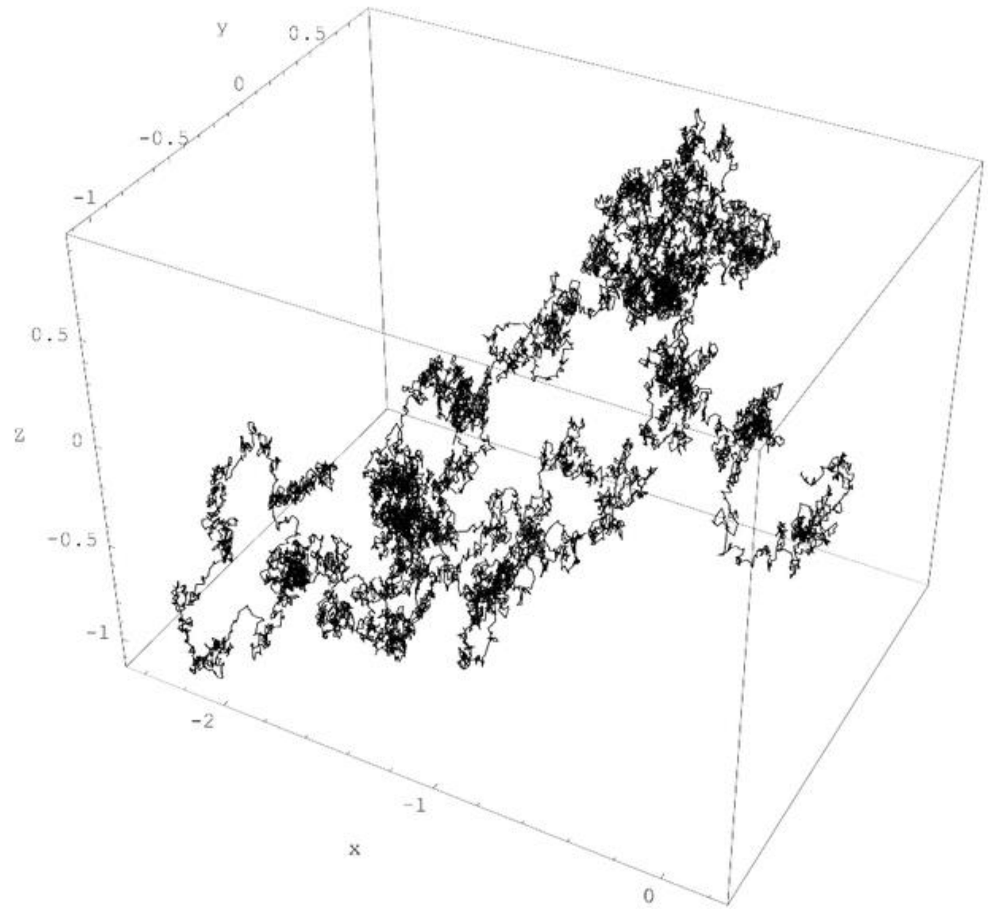


Using DLA & percolation

- Distribution of galaxies
- Microcosm
- Porous media
- Clouds, rainfall areas
- Simulation of growth
- Crystals

Brownian motion

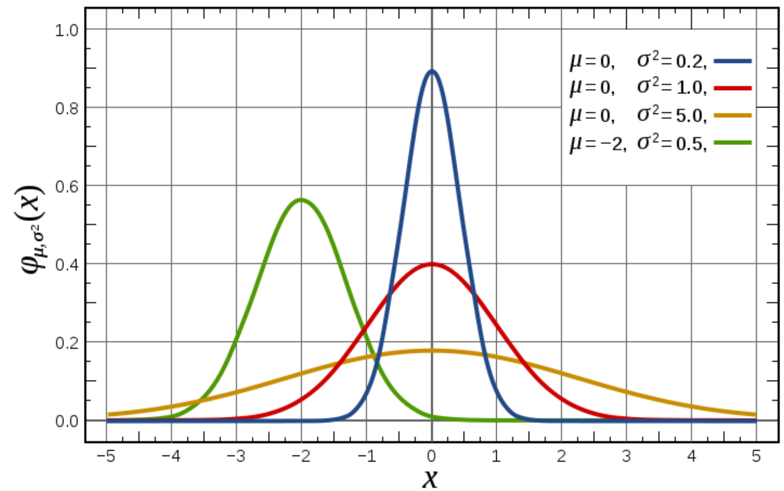
- “Chaotic” movement of particles
- Related to Gaussian distribution
- Statistically self-similar fractal
- Base for other statistical fractals



Gaussian distribution

- Probability density function

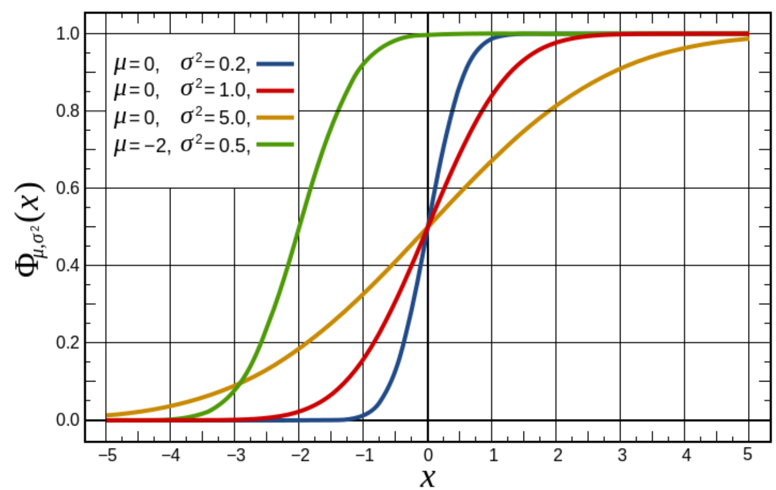
$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



- Cumulative distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad x \in \mathbb{R}.$$

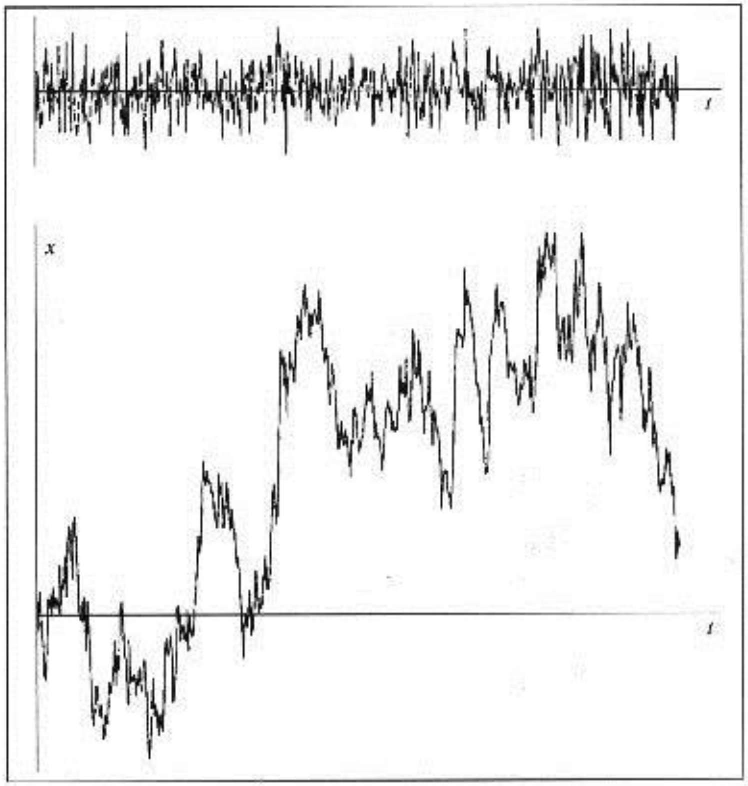
$$F(x;\, \mu, \sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \quad x \in \mathbb{R}.$$

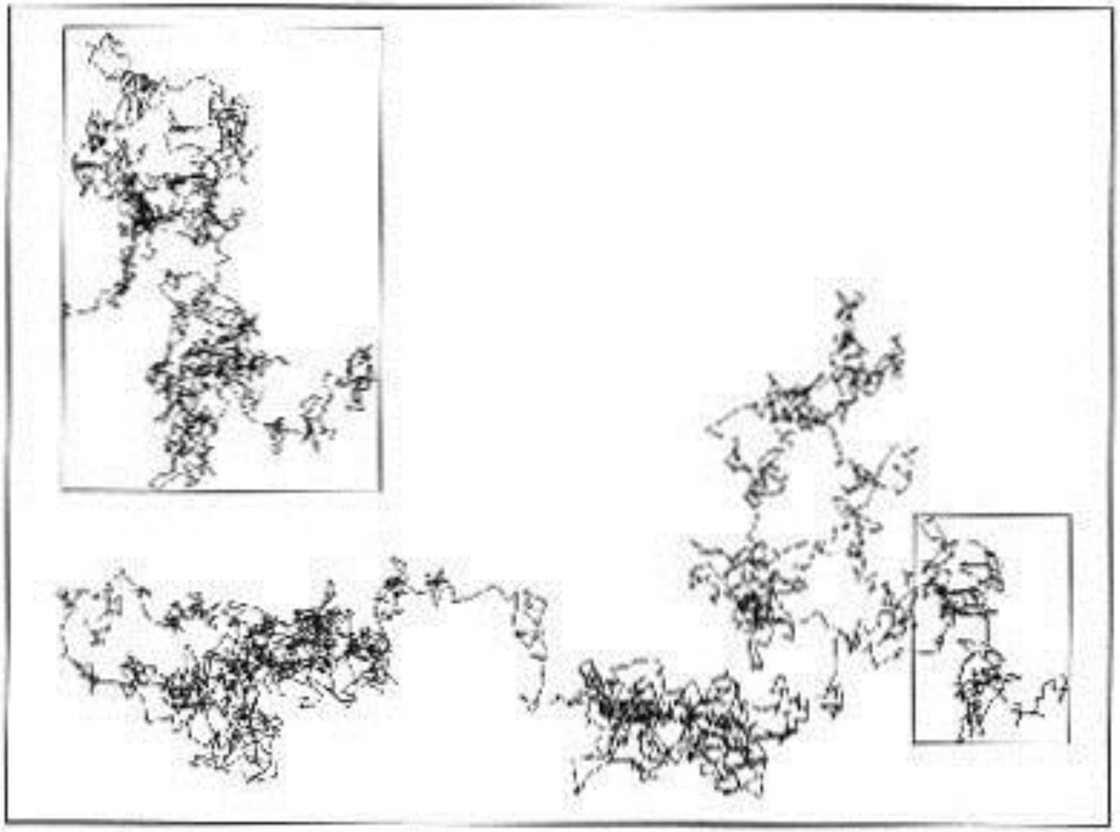


Simulation of BM

- 1D simulation – Wiener process
  - $W_0 = 0$, $W_t - W_s \sim N(0, t-s)$
- Proceeding in t with uniform steps
  - $X(0)=0$, $X(k)=D_1+\ldots+D_k$; $k=1,2,3,\ldots$
  - Using Gaussian random numbers (expected value 0, variance 1) as displacement D_k
- Scaling:
  - $V_t=1/\mathrm{sqrt}(c)W_{ct}$ is another Wiener process
  - in *x* 2 times, in *y* sqrt(2) times
  - $X(t+dt)-X(t)$ and $1/r^{0.5}(X(t+r.dt)-X(t))$ are statistically equivalent
- $X(t+dt) = X(t) + v \bullet dt^{0.5} \bullet N(0,1)$
  - v – speed of particle

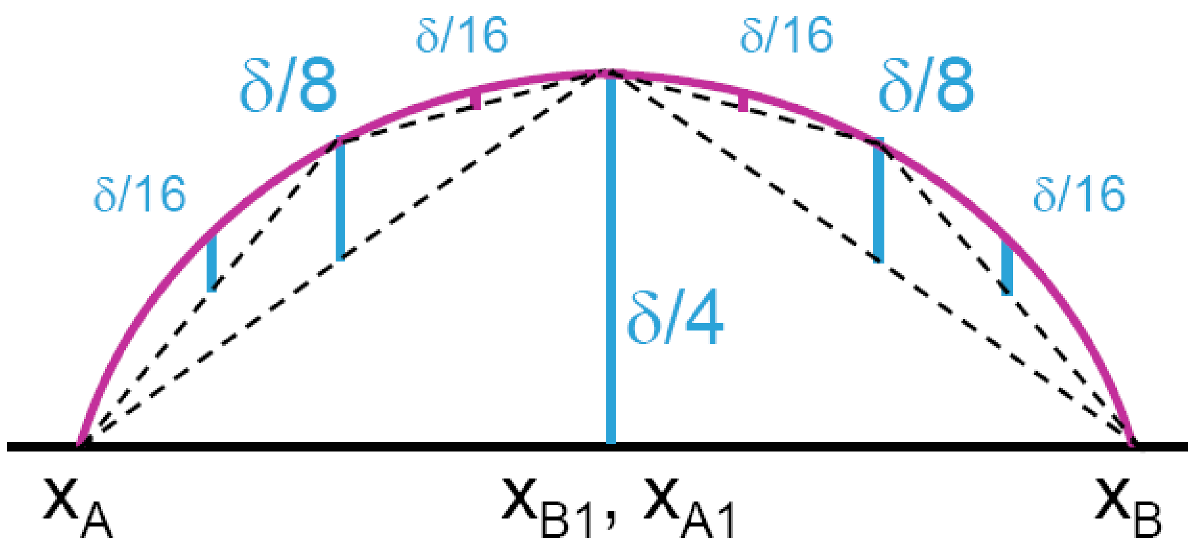
Simulation of BM



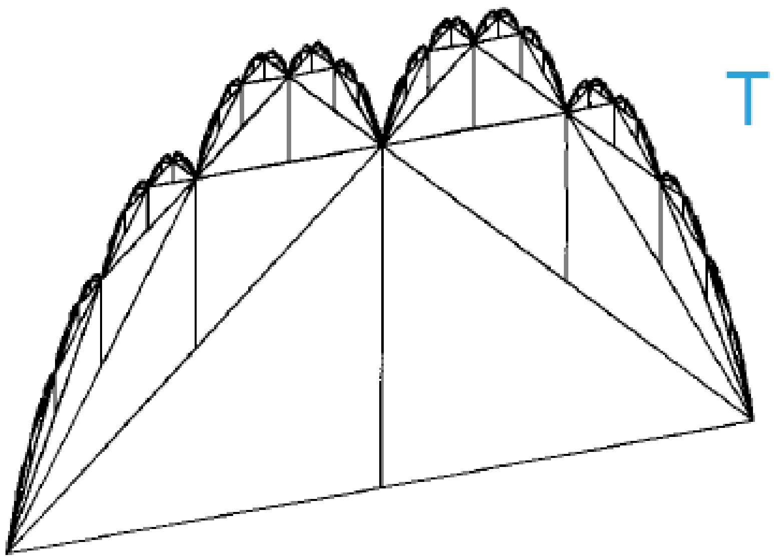


Midpoint displacement

- Construction of parabola $P(x) = a - bx^2$, $b > 0$
- Archimedes



Midpoint displacement



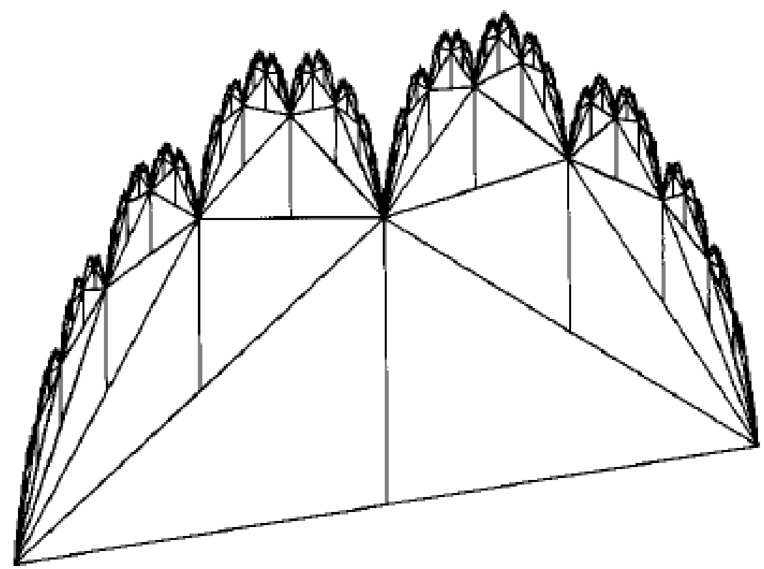
Takagi curve

$$\text{blanc}(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n},$$

$$s(x) = \min_{n \in \mathbf{Z}} |x - n|$$

$$T_w(x) = \sum_{n=0}^{\infty} w^n s(2^n x)$$

Landsberg curve



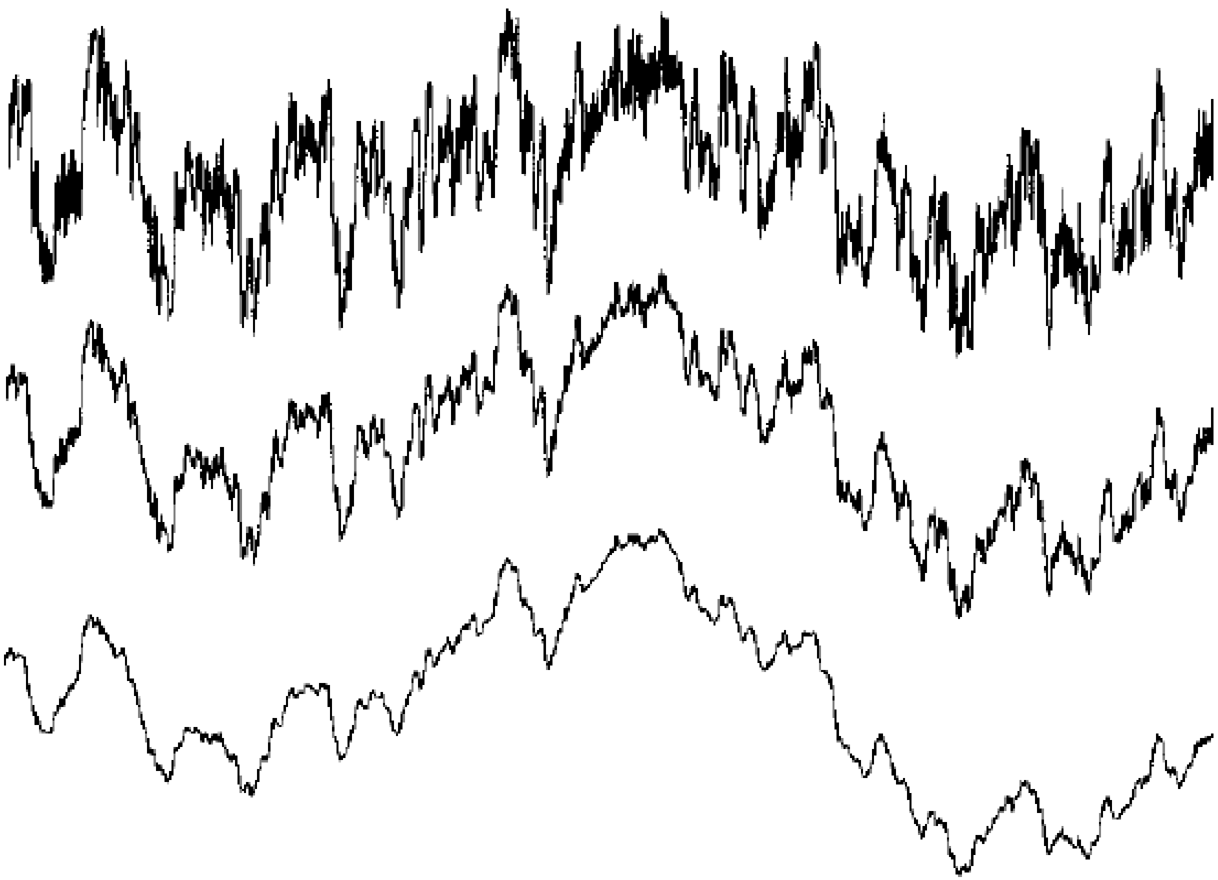
Midpoint displacement

- In each stage for each line displace midpoint in y-direction with Gaussian random number multiplied by scale
- X(0)=0; X(1)=GRN
- $X(1/2)=(1/2)*(X(0)+X(1))+D_1/sqrt(2)$
- Recursive algorithm

Hurst exponent

- Creating Fractal Brownian Motion (FBm)
- Can be generalized with parameter H
- In i-th step, multiply Gaussian random number by 2^{-Hi}
- H – Hurst exponent, $0<H<1$
- Curves dimension $D = 2-H$
- Surfaces dimension $D = 3-H$

Exponents and dimensions



H = 0.2

D = 1.8

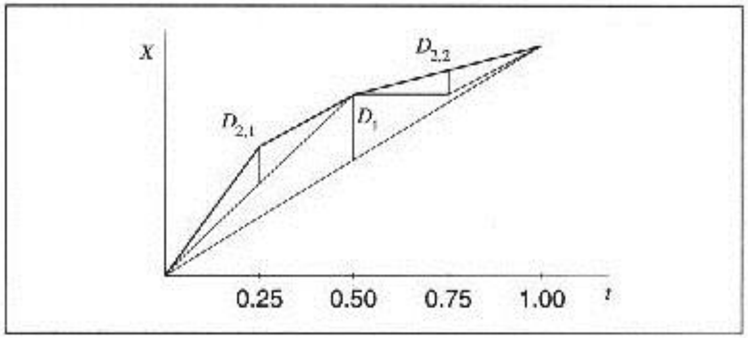
H = 0.5

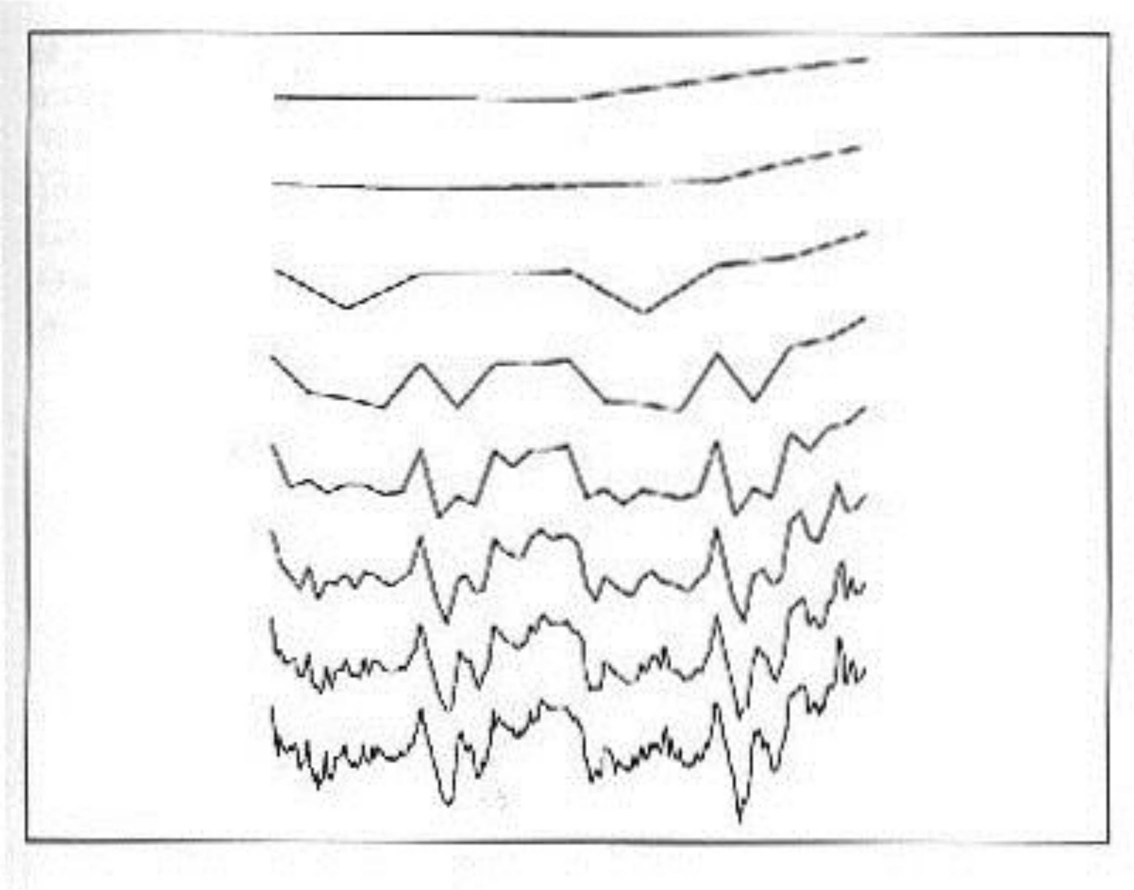
D = 1.5

H = 0.8

D = 1.2

Midpoint displacement 2

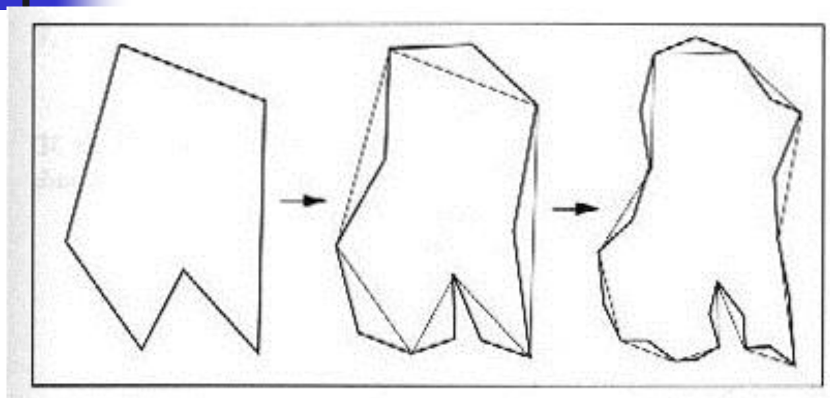


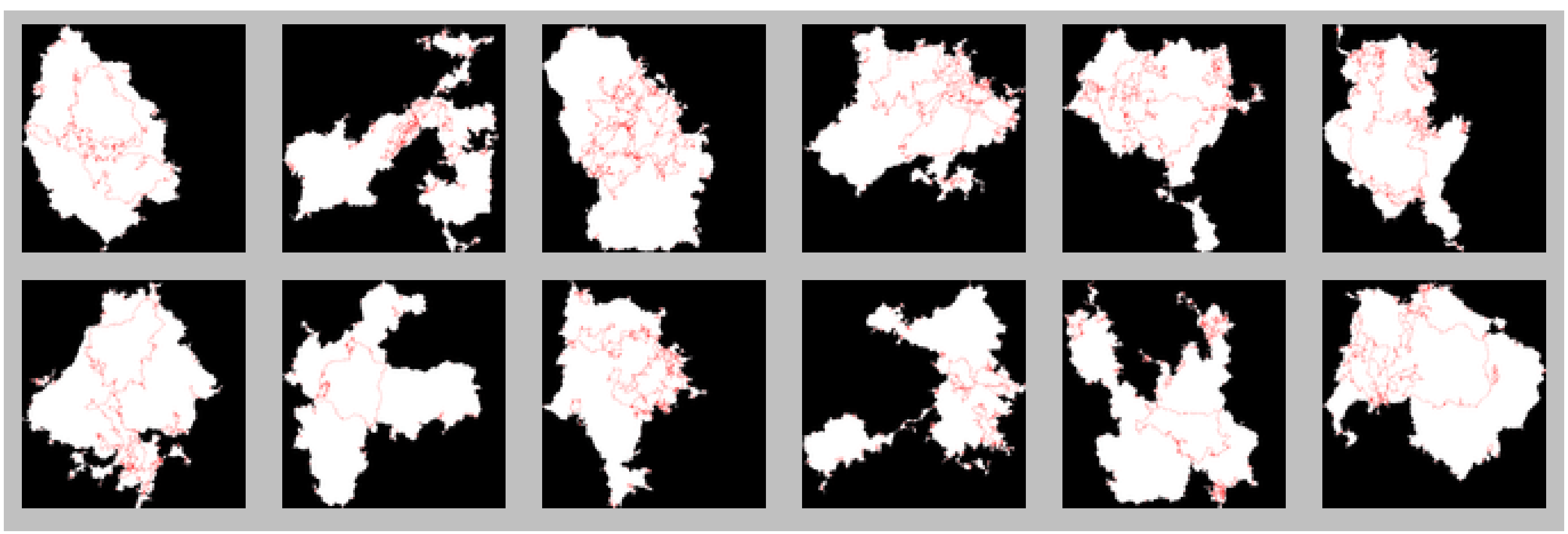


Midpoint displacement ext.

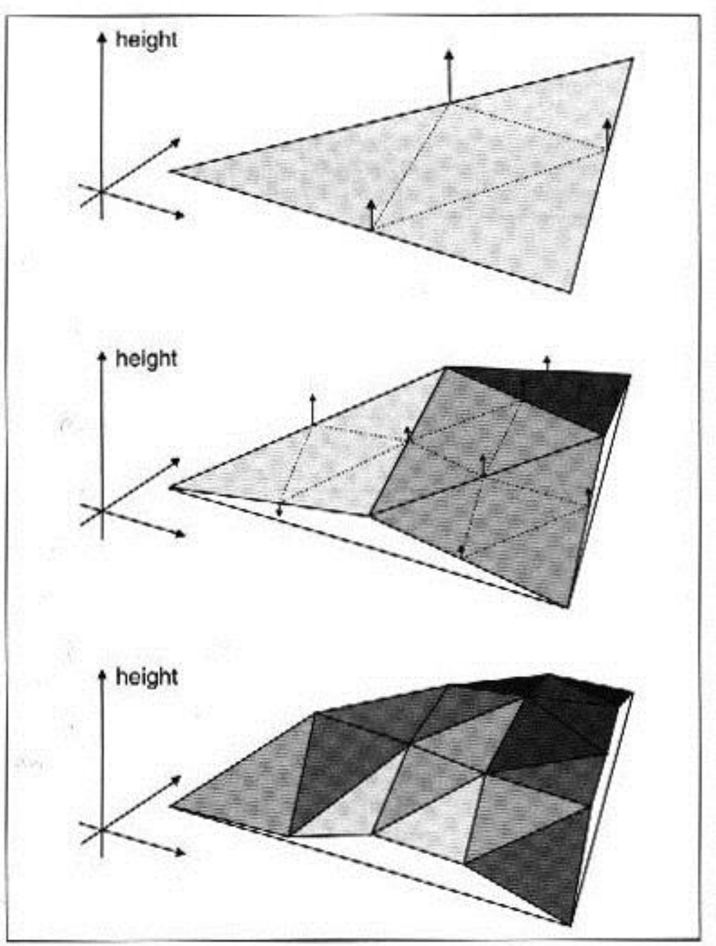
- Modeling natural objects
- Coastline = initial closed polygon,
- Landscape = extension to 2D, dividing triangles or squares
- Fake clouds = colored height map
- True clouds = map of points above threshold

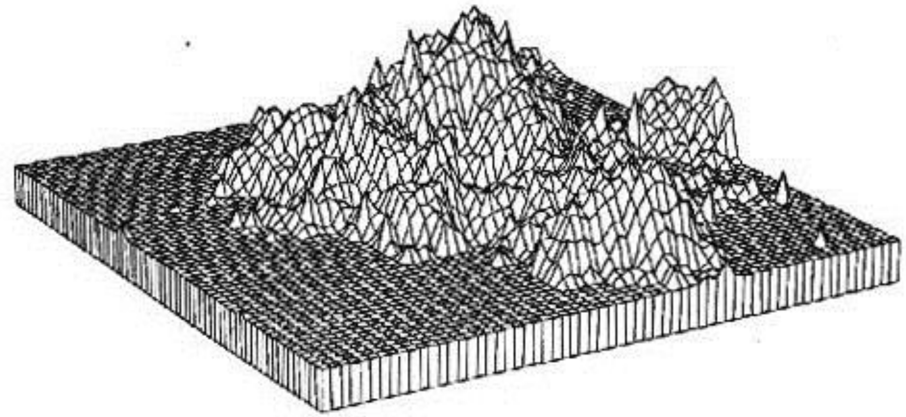
Coastline

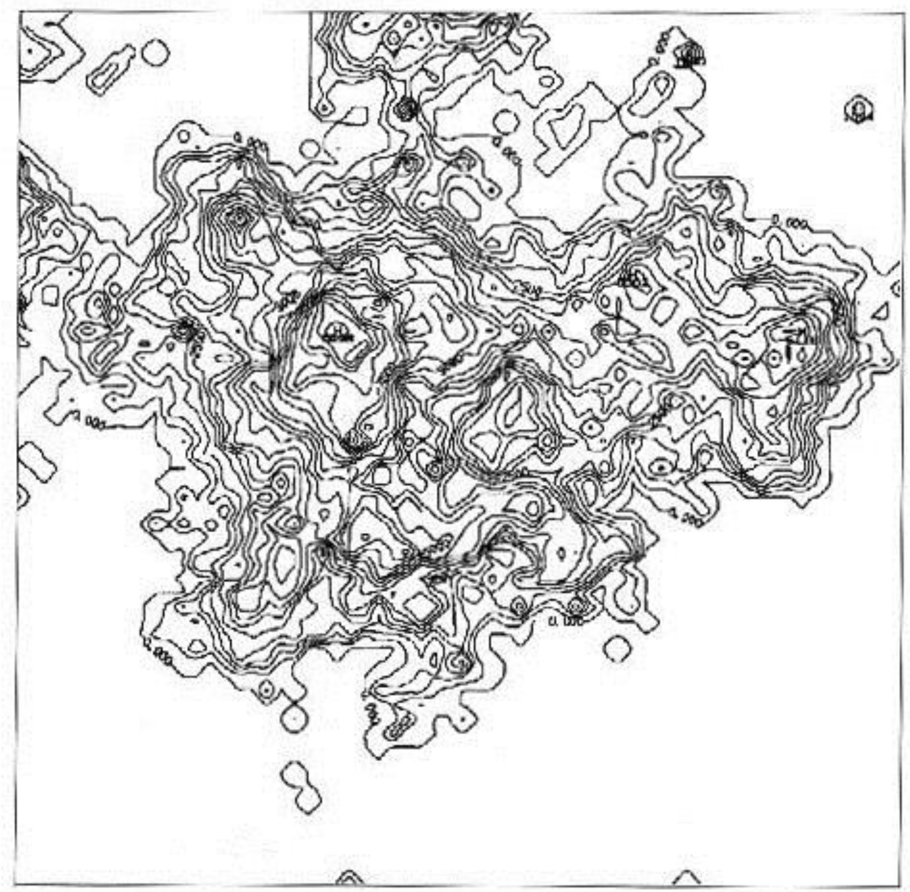




Landscapes

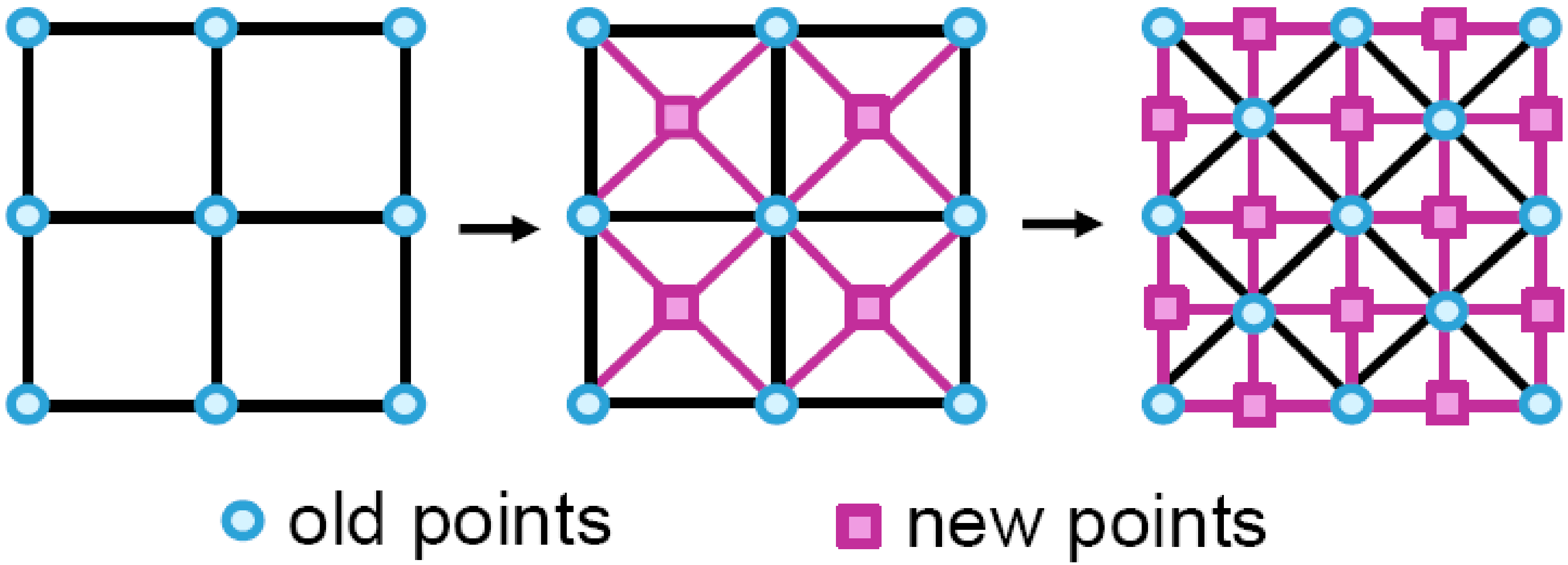






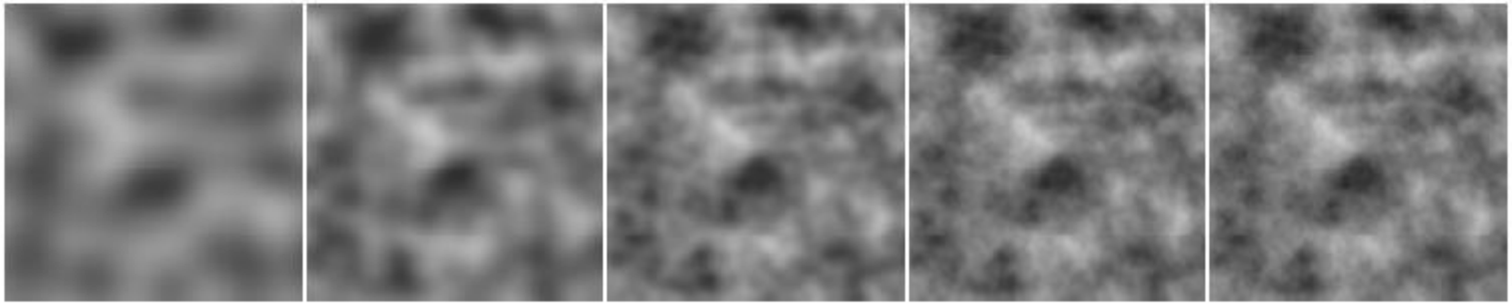
Landscapes

- Better method – subdividing square
- Diamond-square algorithm



Improvements

- Merging of different terrains with several Hurst exponents
- Adding fractal noise to smooth terrain
- Spectral Synthesis Method – remove high frequencies using Fourier transform



Visualization

- Mostly visualization of height map
- Coloring: Mapping aerial textures, using height for color
- Acceleration structures – quadtrees
- HW support – tesselation, vertex shaders – displacement mapping
- Raytracing – subdivided are only parts of terrain that are hit by some ray

Landscapes

Professional landscape

Generating clouds

- 2D clouds = height map
- Draw the height field as color map with different transparency based on height

End

End of Part 8