Fractals

Part 9 : Chaos & Strange attractors

Martin Samuelčík
Department of Applied Informatics

Chaos theory

- Based on Poincare work
- Mathematically evaluated, Deterministic
- Still sometimes hard to compute properties
- Lots of sudden unpredictable changes
- Behaviour of iterations (solution of differential equations)
- Butterfly effect

Principles of chaos theory

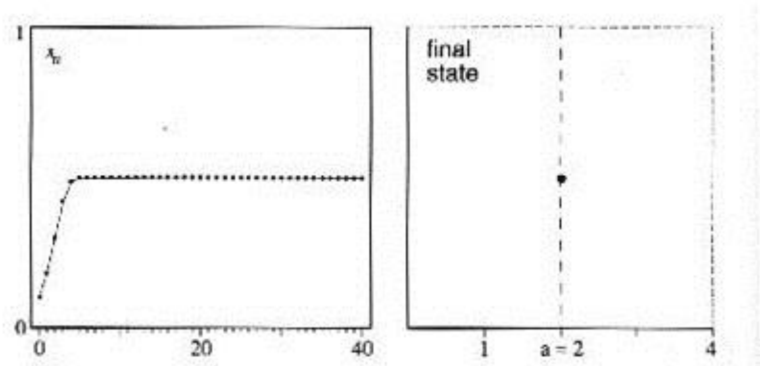
- Sensitivity, stretching - sensitive to initial conditions
- Mixing – from state to any state
- Folding – states still remain in some closed set
- Periodic points – state is periodically repeated
- Mixing and periodic points -> sensitivity

Quadratic iterator

- Simple quadratic equation
- $x_{n+1} = a x_n(1-x_n)$
- Simulation of population growth
- Observing iterations, for any *a* we can find different behavior
- a is basically from [0,4]
- Drawing final-state diagram

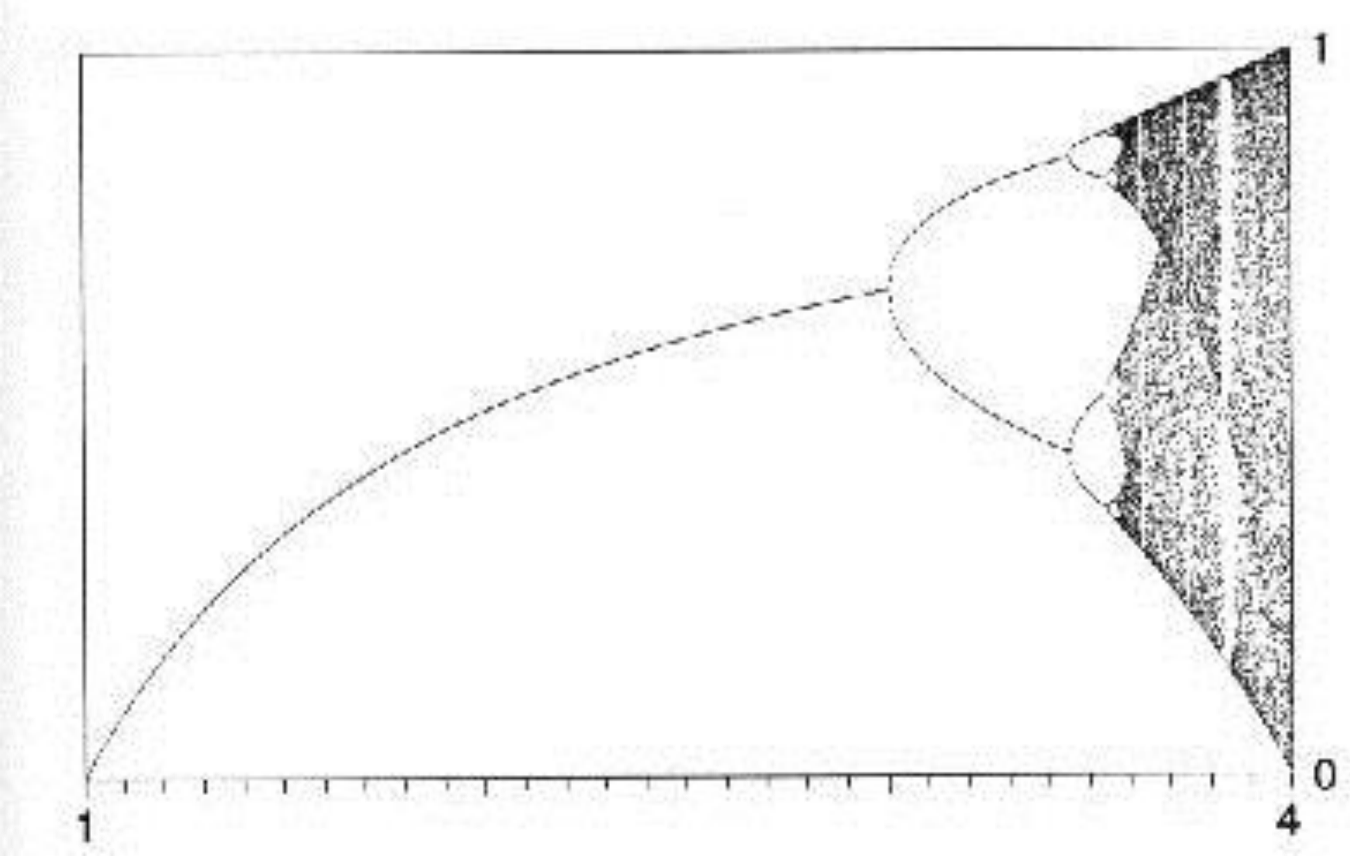
Final-state diagram

- For given *a* compute iterations
- Choose initial value from [0,1]
- Do 200 iterations $x_1, \ldots, x_{200}$
- Drop first 100 iterations
- Plot remaining iterations in the diagram



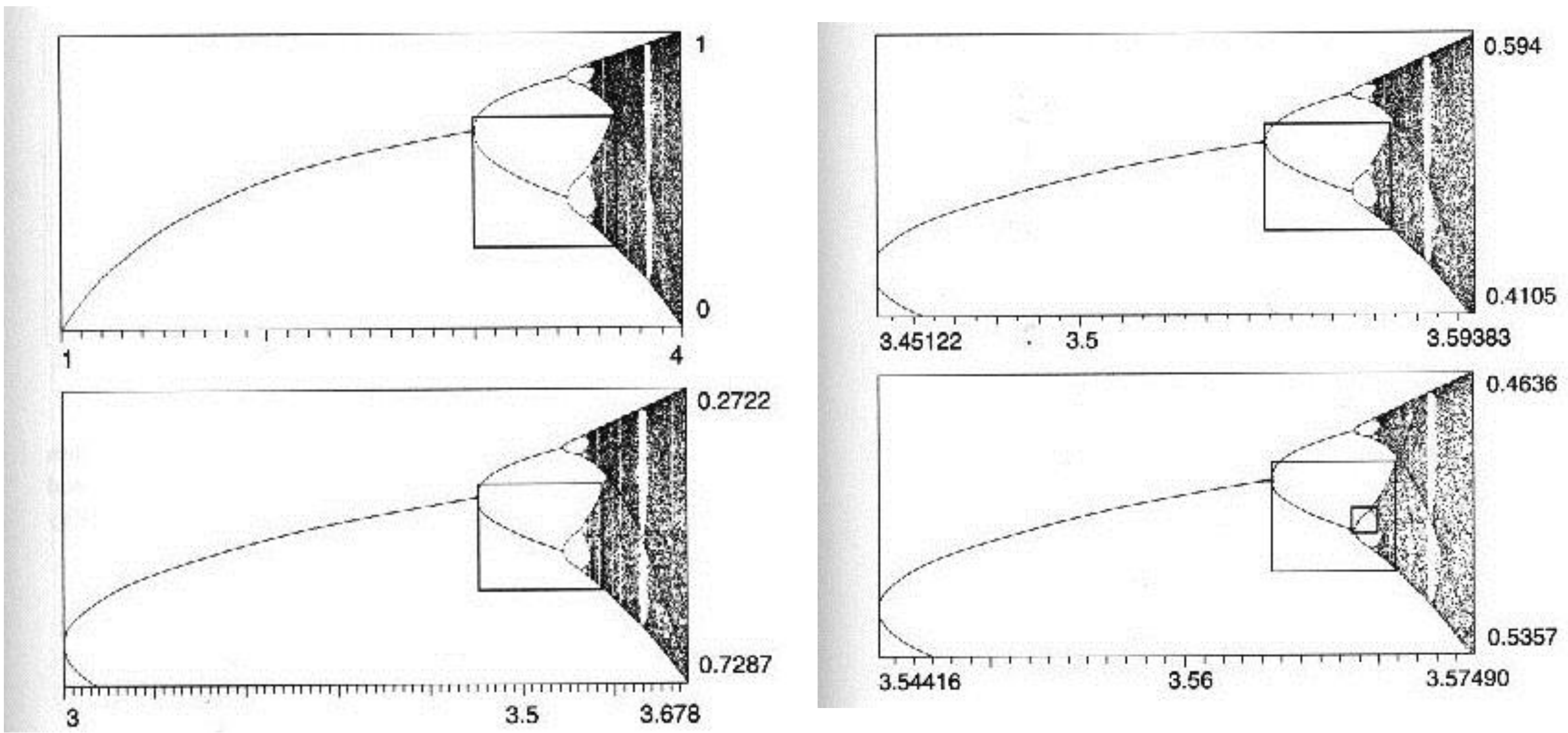
Whole final-state

- Feigenbaum diagram



Period-doubling

- Each branch is bifurcated into two new
- Self-similarity



Feigenbaum point

- Limit of branches in diagram
- Ends period-doubling tree
- Starts area governed by chaos
- $a = s_{inf} = 3,5699456...$
- Feigenbaum constant = ratio of length of two adjacent branches
- $\delta = 4,6692...$

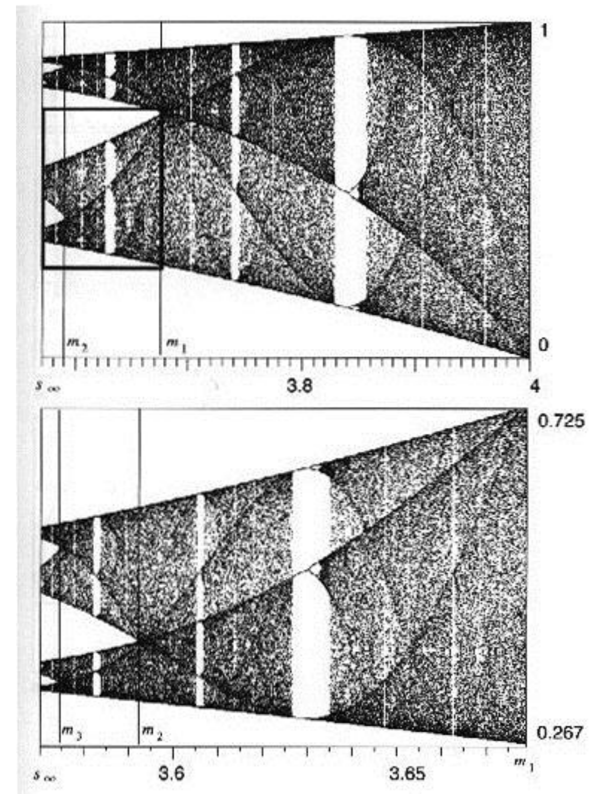
Iterations

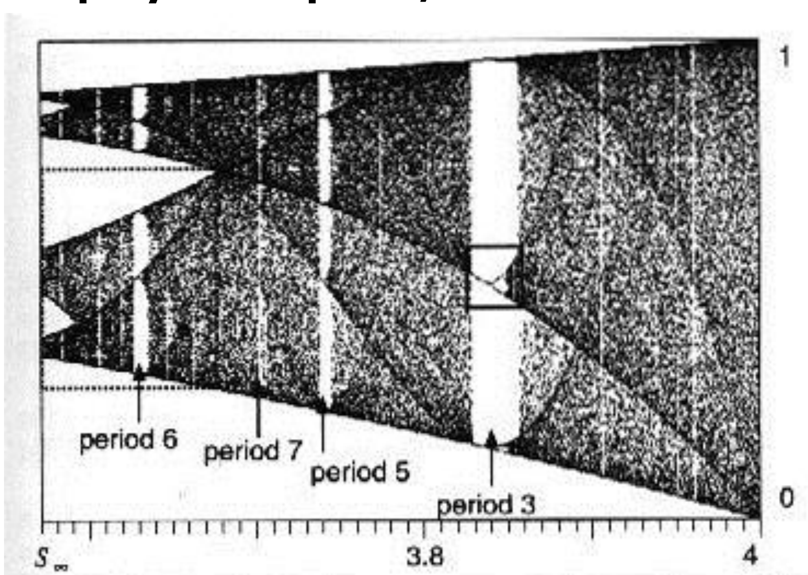
- a in (0,1) - limit is 0
- a in [1,2] – limit is (a-1)/a, approach to limit is quick
- a in [2,3] – limit is (a-1)/a, approach fluctuates around that value
  - a=2 $x_n = \frac{1}{2} - \frac{1}{2}(1-2x_0)^{2^n}$
- a in [3,1+sqrt(6)] – limit oscillates between 2 values, can be computed as roots of cubic function
- a in [3,3.5699…] – period doubling part
- a in [3.5699…,4] – chaos side
  - a=4 $x_{n+1} = \sin^2(2^n\theta\pi)$ $\theta = \frac{1}{\pi}\sin^{-1}(x_0^{1/2})$

Periodic for rational θ

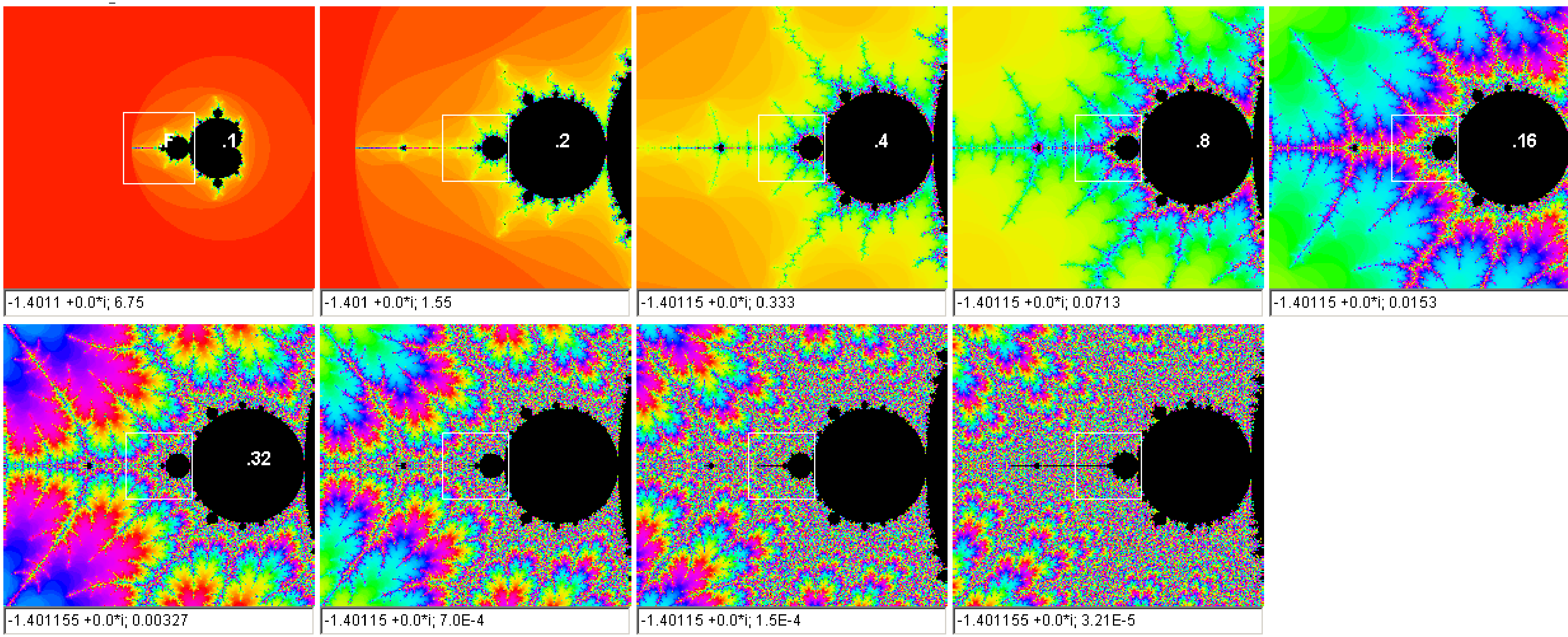
Observing the chaos side

- There are spitted bands
  at the beginning, contains δ
- There are period windows
  (empty stripes, islands of stability)





Mandelbrot <-> Feigenbaum



Differential equations

- Numerical solutions

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

- Euler method

$$t_n = t_0 + nh \qquad y_{n+1} = y_n + hf(t_n, y_n).$$

- Runge–Kutta method (fourth-order)

$$y_{n+1} = y_n + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$t_{n+1} = t_n + h$$

$$\begin{aligned} k_1 &= hf(t_n, y_n), \\ k_2 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1), \\ k_3 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2), \\ k_4 &= hf(t_n + h, y_n + k_3). \end{aligned}$$

Dynamical systems

- Based on real world observations or theoretical computations
- Solution of dynamical system or iterated transformations
- Plots strange shapes in 2D, 3D
- Attractors of such systems

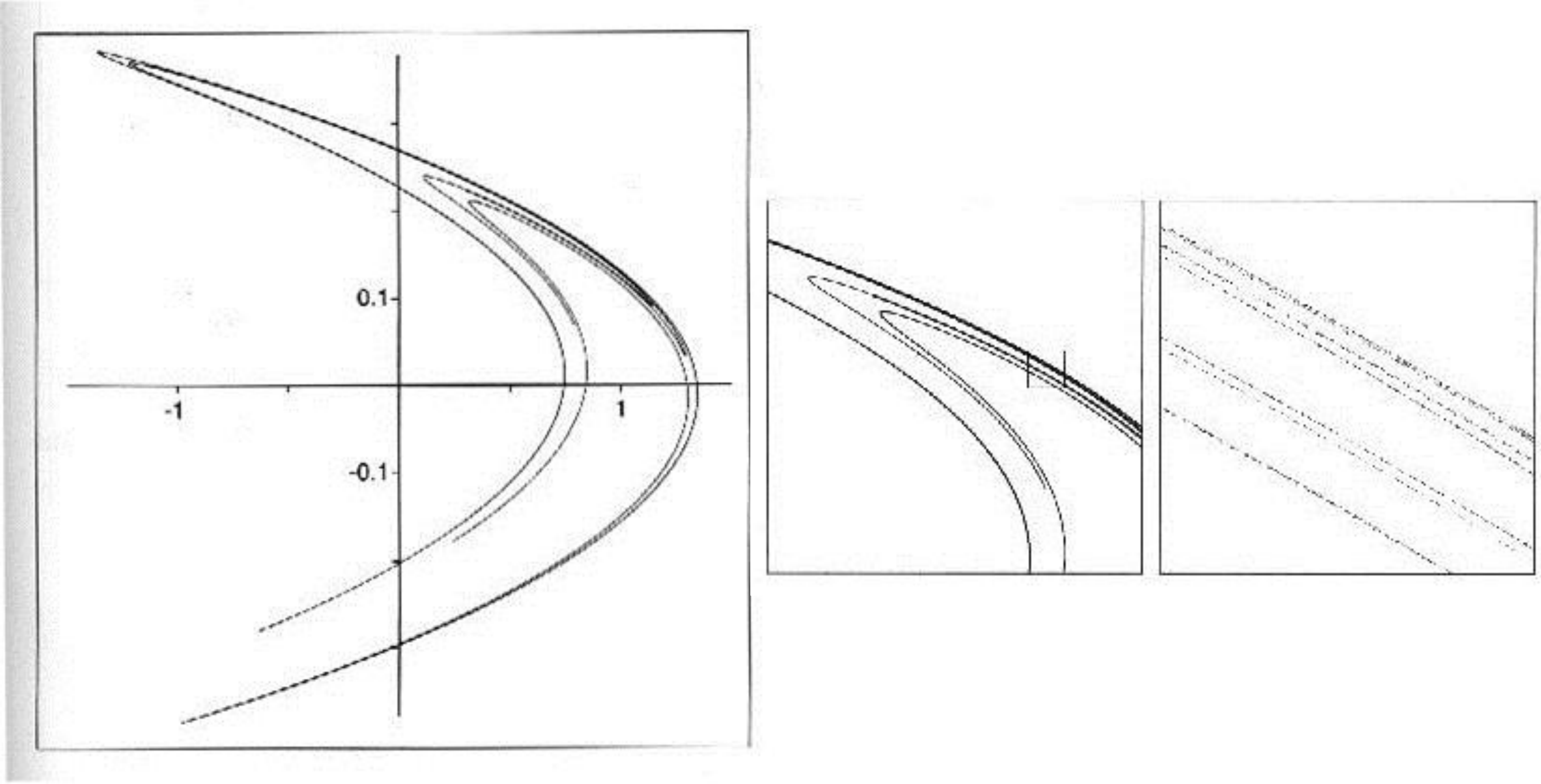
Dynamical systems

- Starting conditions x_0, y_0, z_0
- Number of iterations
- Parameters $p_0..p_n$
- for (n=0 to number of iterations) do
  - $x_{n+1}=f_1(x_n, y_n, z_n, p_0...p_n)$
  - $y_{n+1}=f_2(x_n, y_n, z_n, p_0...p_n)$
  - $z_{n+1}=f_3(x_n, y_n, z_n, p_0...p_n)$
  - paint_point(x_{n+1}, y_{n+1}, z_{n+1})

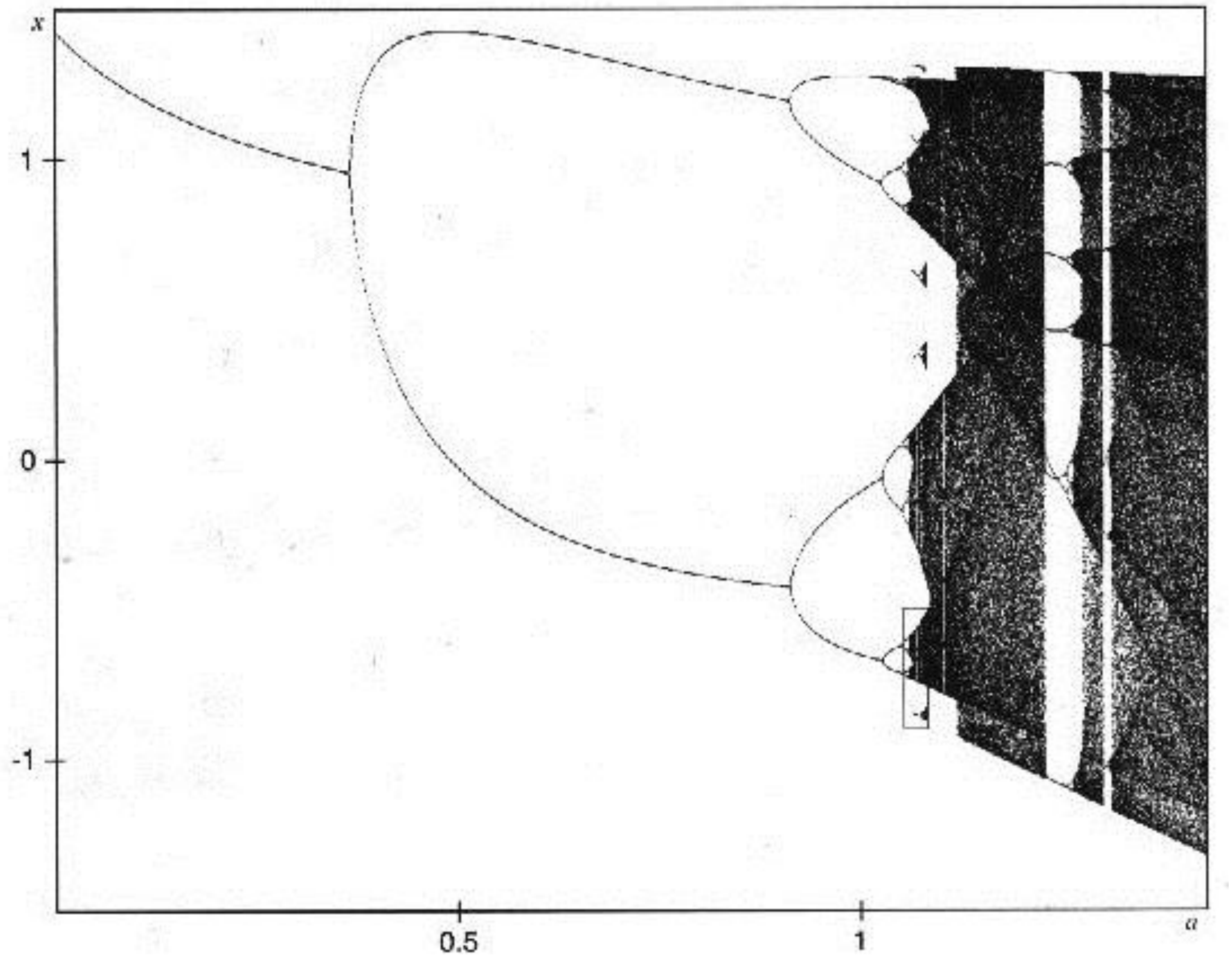
Henon's attractor

- French astronomer Michel Henon
- $H(x,y) = (y+1-ax^2,bx)$
- Computing orbits of transformation
- a=1,4;b=0,3
- Trapping region R=quadrilateral for starting points

Henon’s attractor

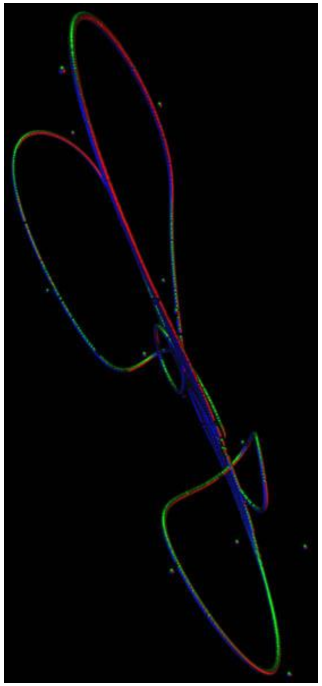
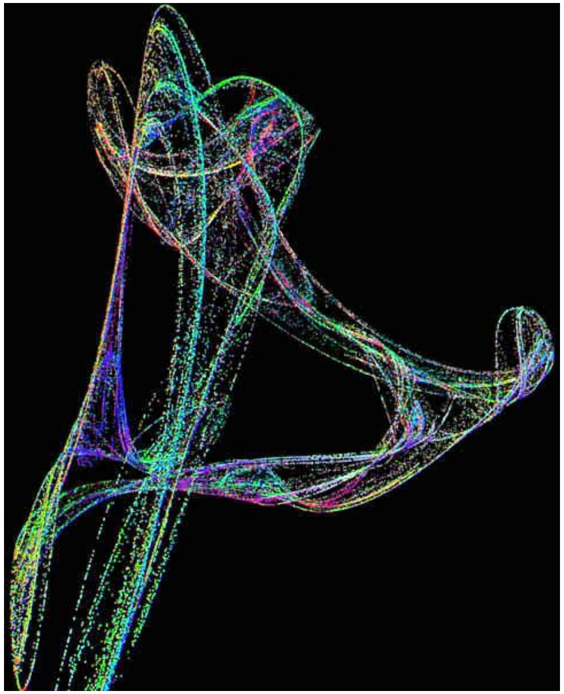
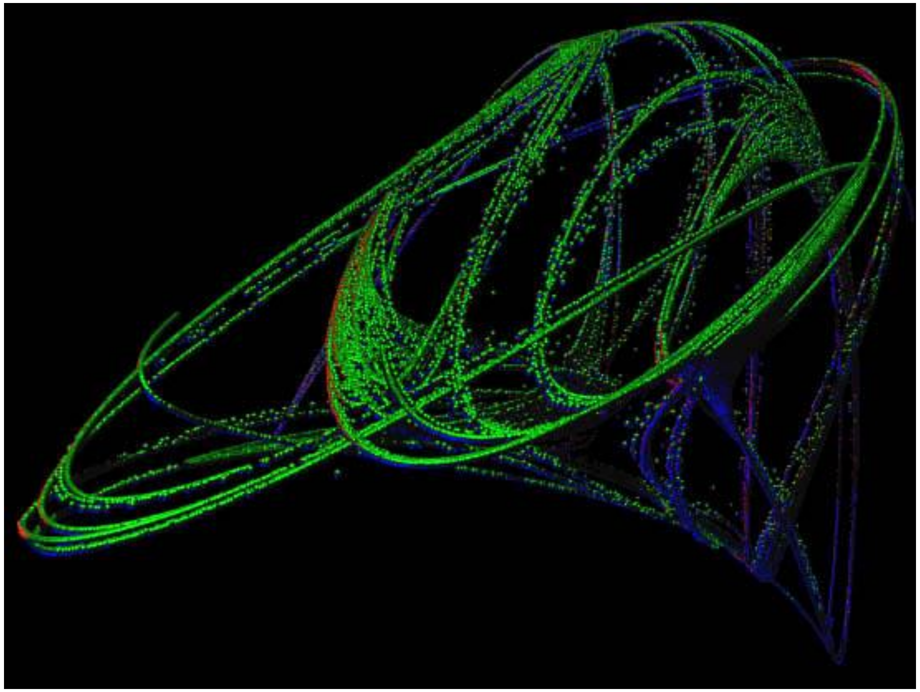


Feigenbaum scenario



Generalized quadratic maps

$$
\begin{aligned}
x_{n+1} &= a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n y_n + a_5 y_n + a_6 y_n^2 \\
y_{n+1} &= a_7 + a_8 x_n + a_9 x_n^2 + a_{10} x_n y_n + a_{11} y_n + a_{12} y_n^2
\end{aligned}
$$

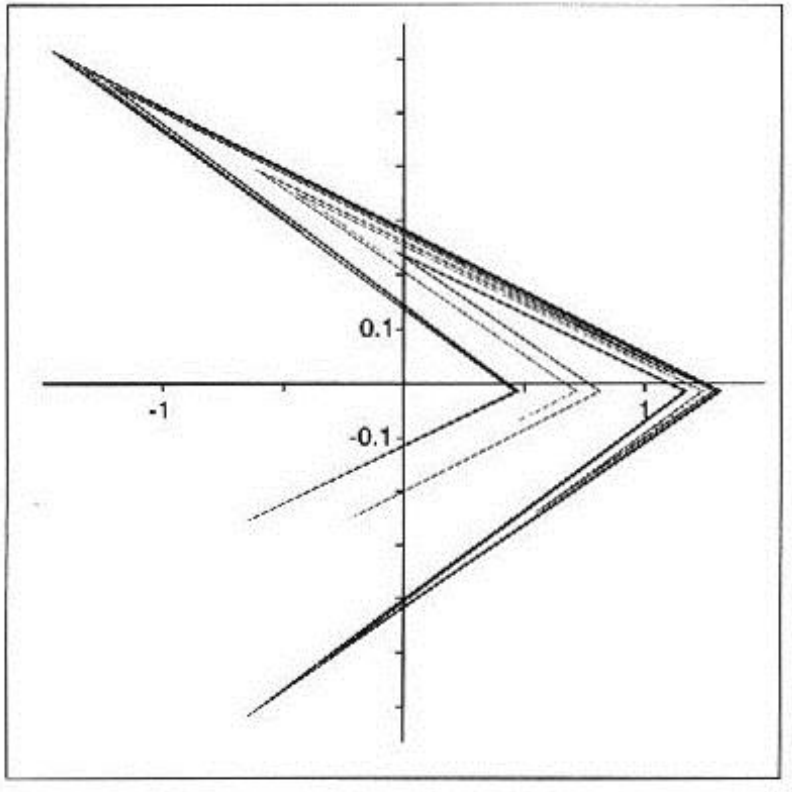




Strange attractors

- A-chaotic and strange attractor
- R-trapping region
- 1. R is neighborhood of A
- 2. Orbits from R are sensitive on initial conditions
- 3. Attractor has a fractal structure
- 4. A cannot be split into two attractors

Lozi's strange attractor

- $H(x,y)=(1+y-a|x|,bx)$
- $a=1,7;b=0,5$

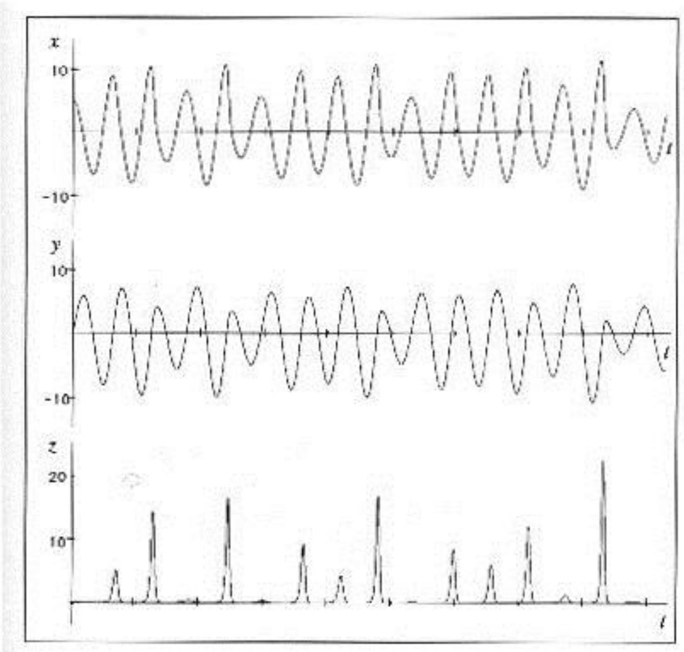


Rossler attractor

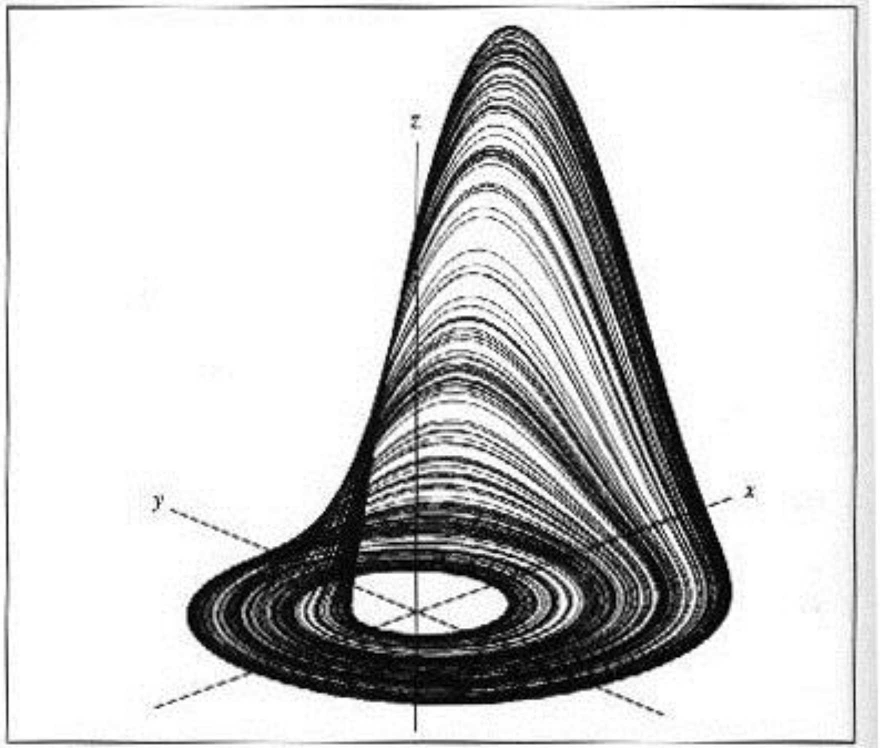
- System of differential equations
- $x' = -(y+z)$
- $y' = x + ay$
- $z' = b + xz - cz$
- Elementary geometric construction of chaos in continuous systems
- $a = b = 0,2$

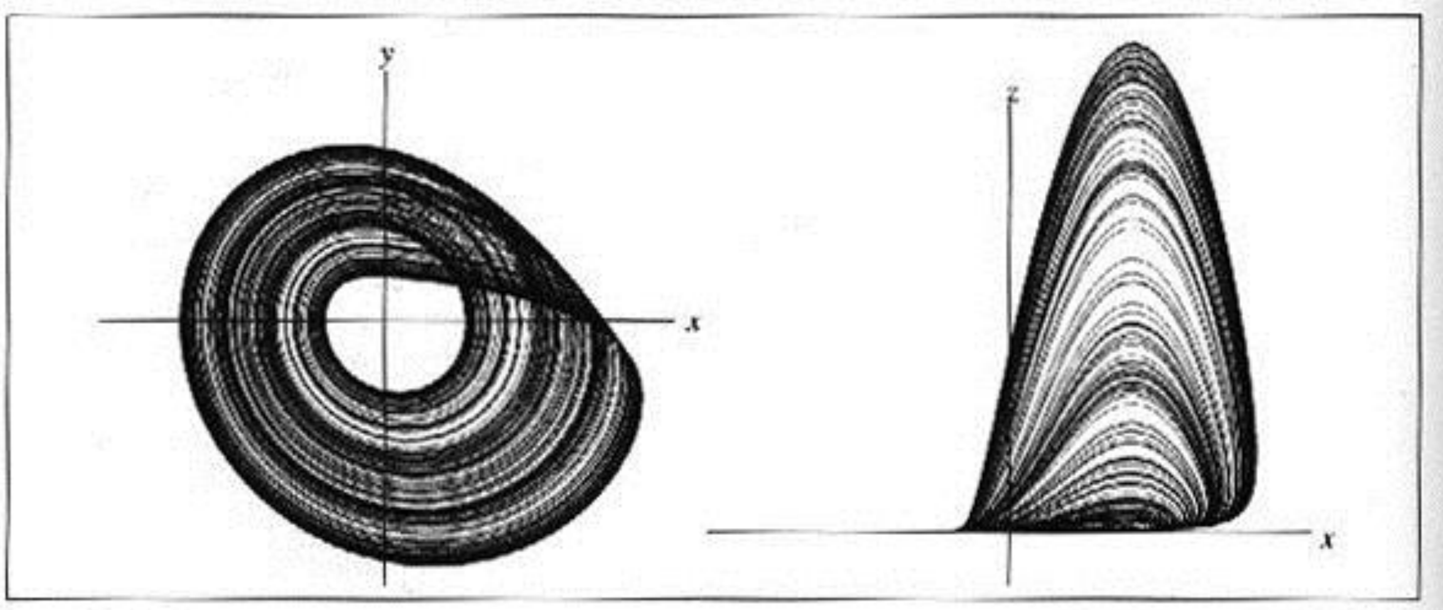
Rossler attractor 2

- Equation is numerically evaluated
- Solution is trajectory with time parameter with starting property=initial point coordinates
- c=5,7 ->

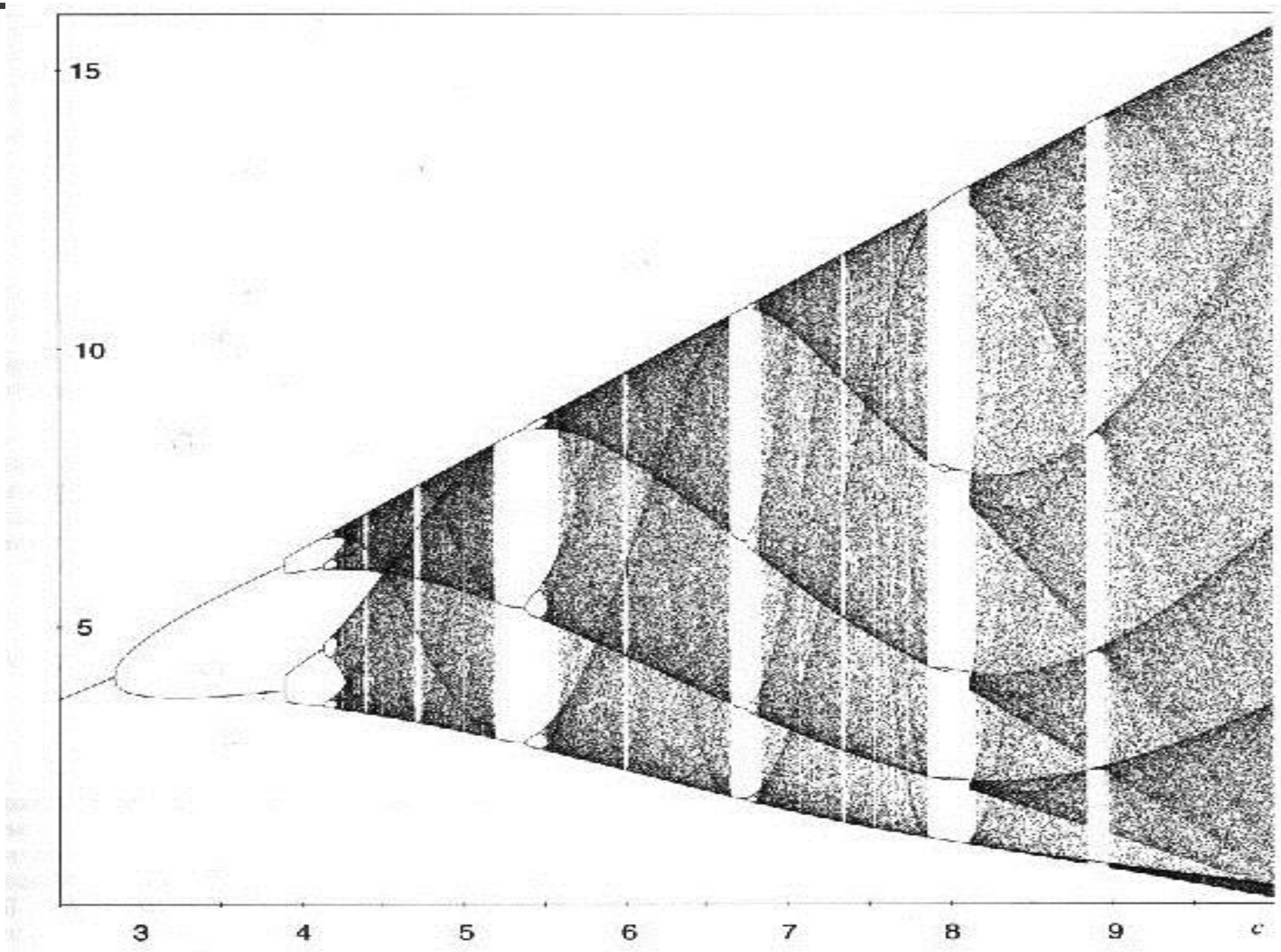


RA 3





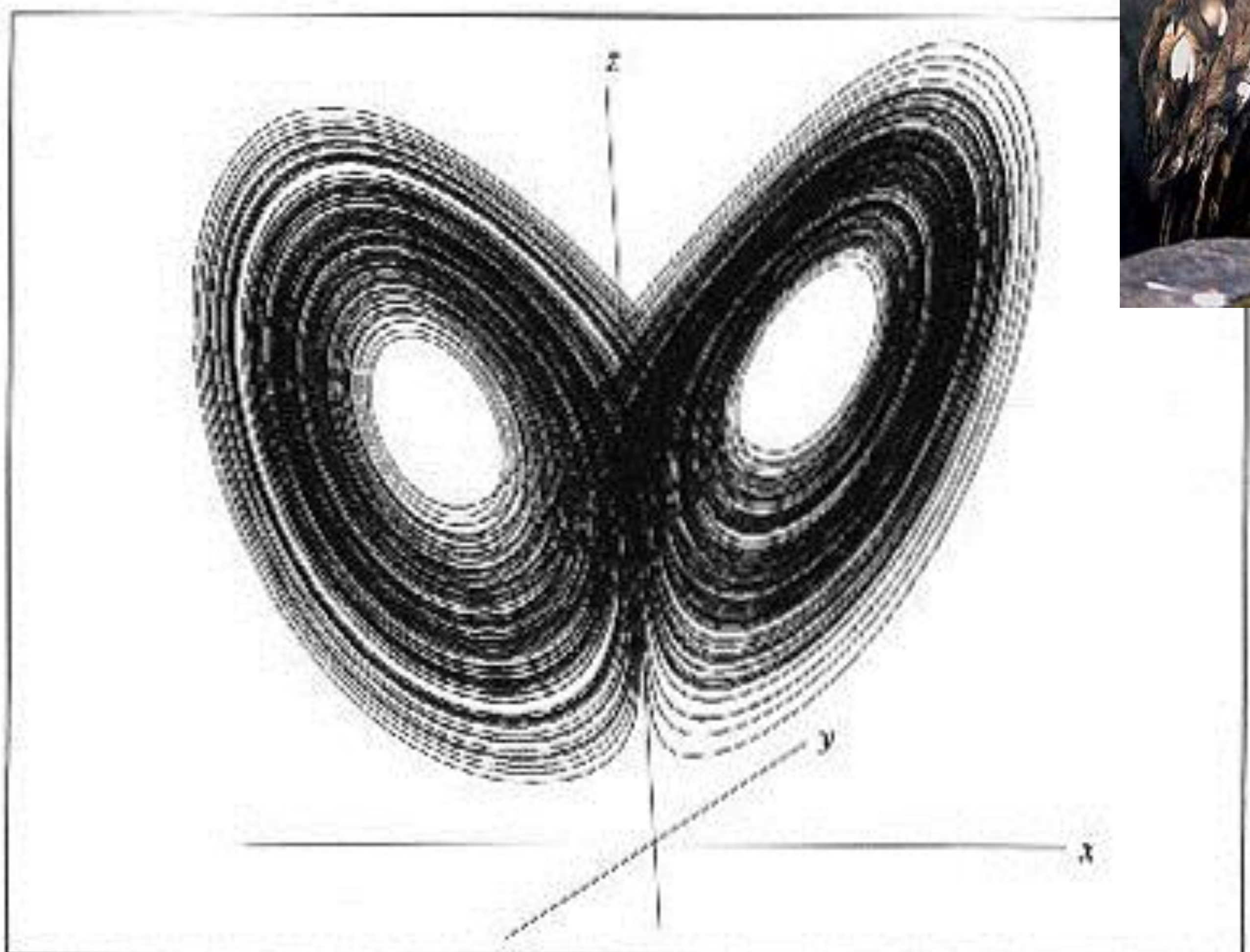
Feigenbaum scenario



Lorenz Attractor

- Again system of differential equations
- $x' = \delta(-x+y)$
- $y' = Rx - y - xz$
- $z' = -Bz + xy$
- $\delta = 10; B = 8/3; R = 28$
- Based on observation of fluid flow and temperature

Lorenz Attractor 2

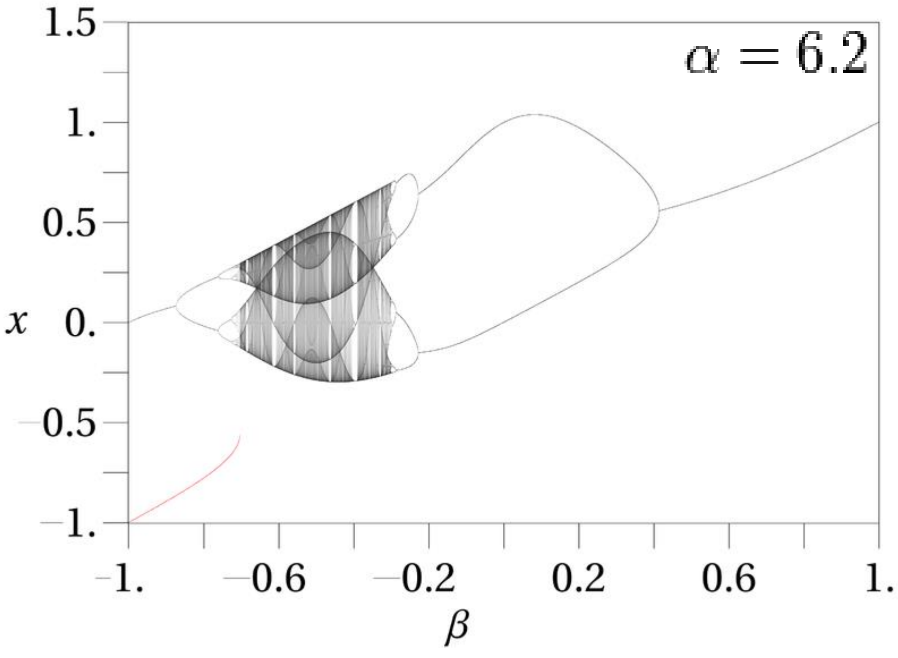


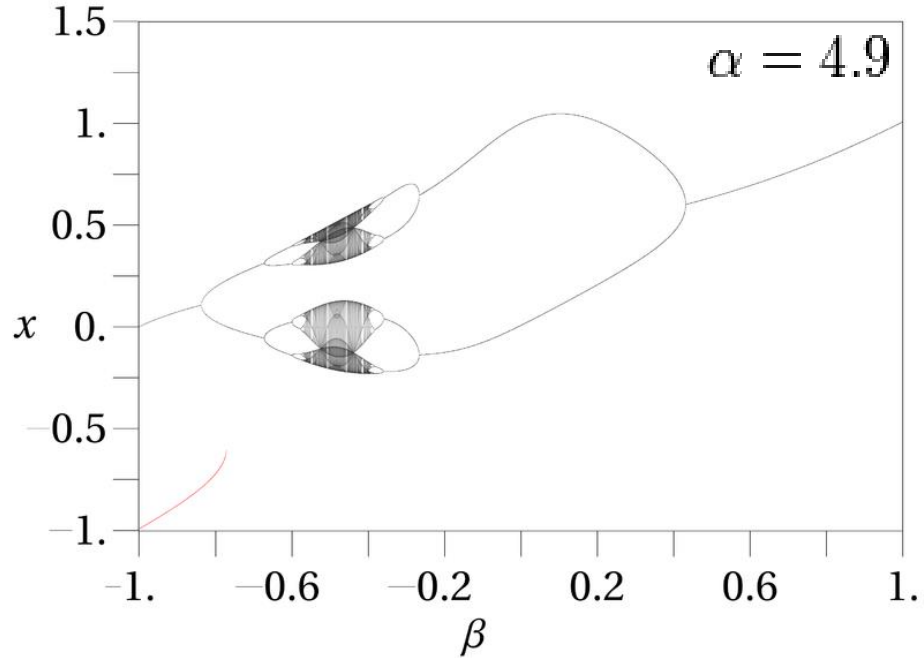
Dimension

- Experiments shows that box-counting method gives non-precise results
- The solution is to compute number of boxes with some connection to box size with numerical parameters
- Strange attractors
  - Lorenz - 2.06 ± 0.01
  - Hénon - 1.261 ± 0.003
  - Feigenbaum (a = 3.5699456...) - ±0.538

Gauss Map

- Nonlinear iterated map $x_{n+1} = \exp(-\alpha x_n^2) + \beta,$
- Called also "mouse map"



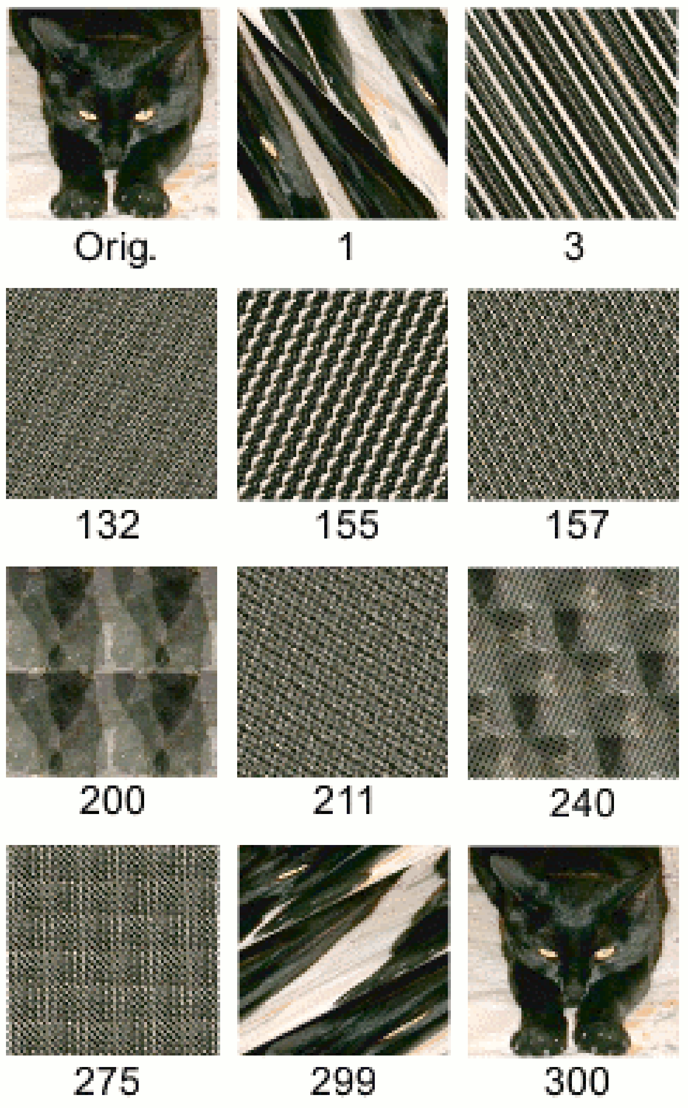


Arnold's cat map

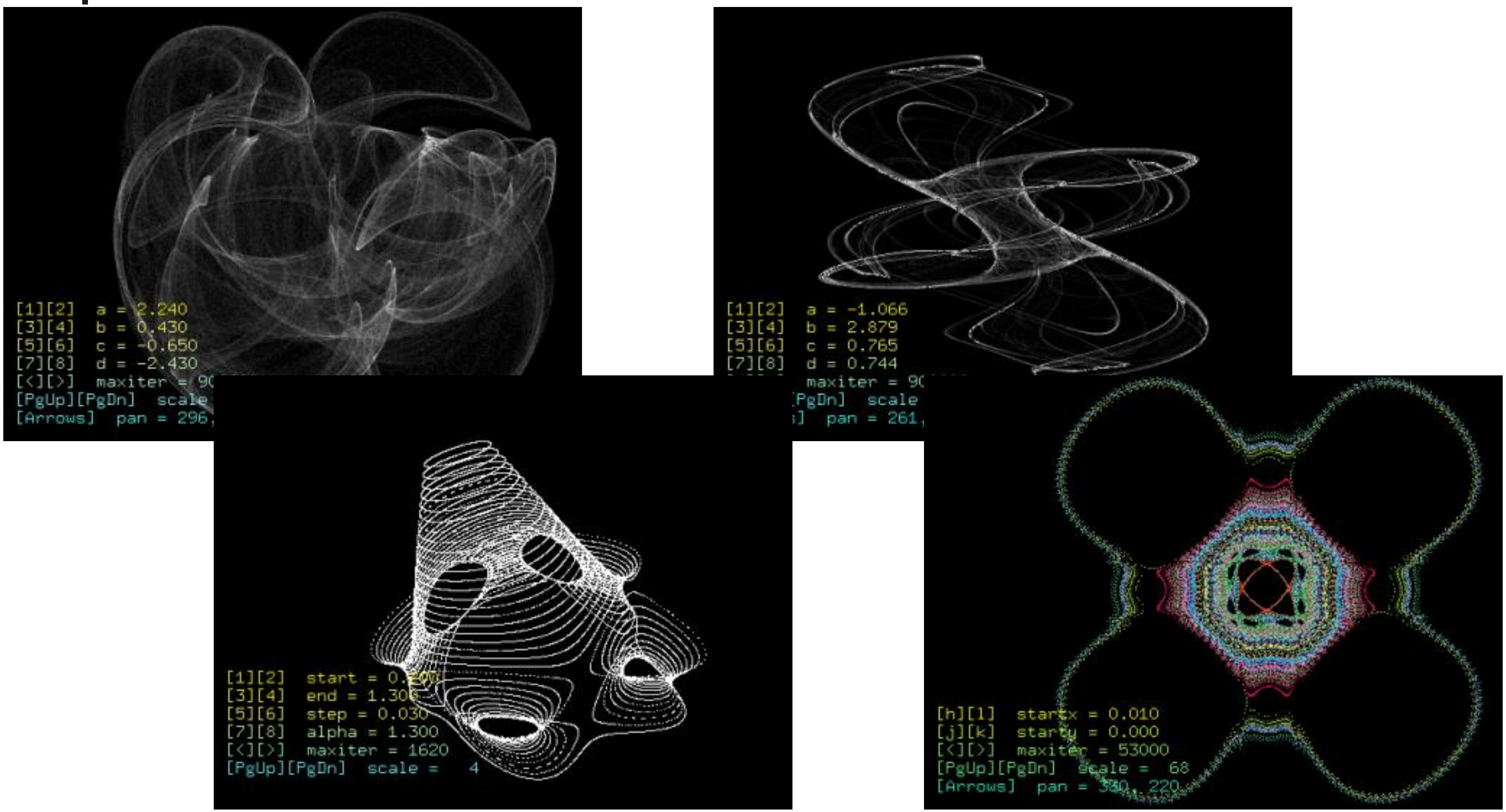
- "Strange" behaviour of image transformation
- Pixels iterations $[p_t, q_t]$

$$q_{t+1} = 2q_t + p_t \mod N$$
$$p_{t+1} = q_t + p_t \mod N$$

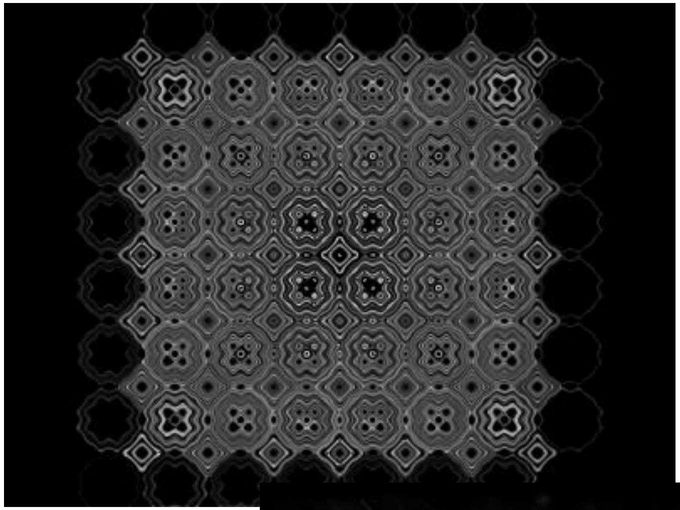
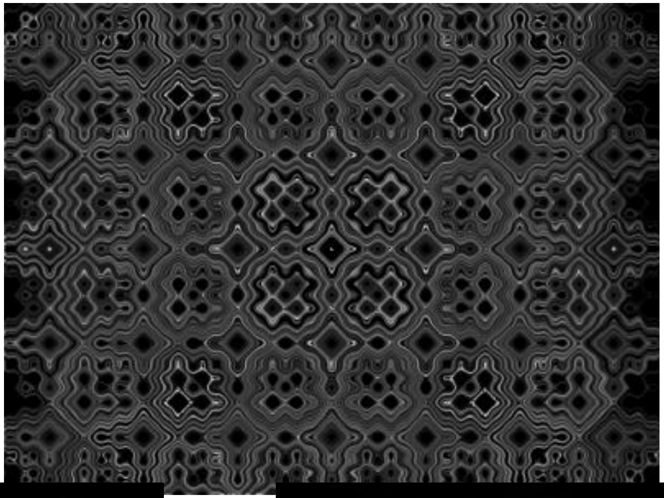
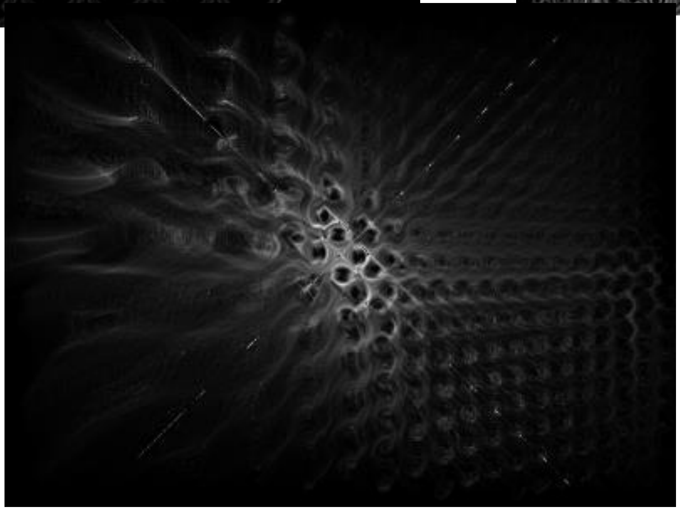
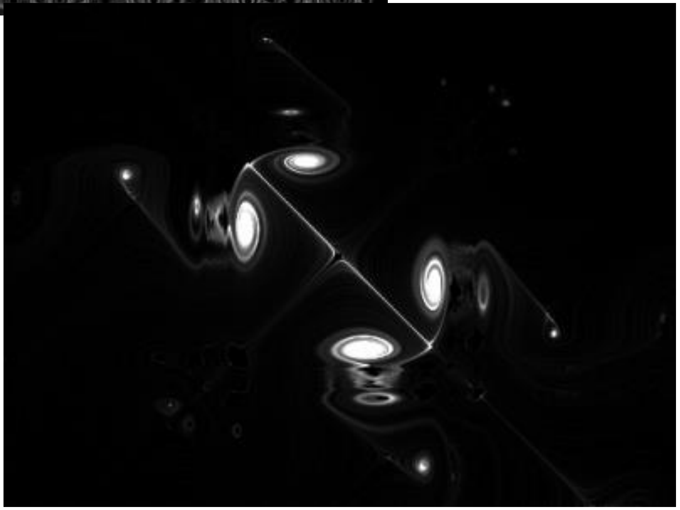
- $0 <= p_t, q_t < N$
- N – width, height of image
- The number of iterations needed to restore the image can be shown never to exceed $3N$



Other Dynamical systems

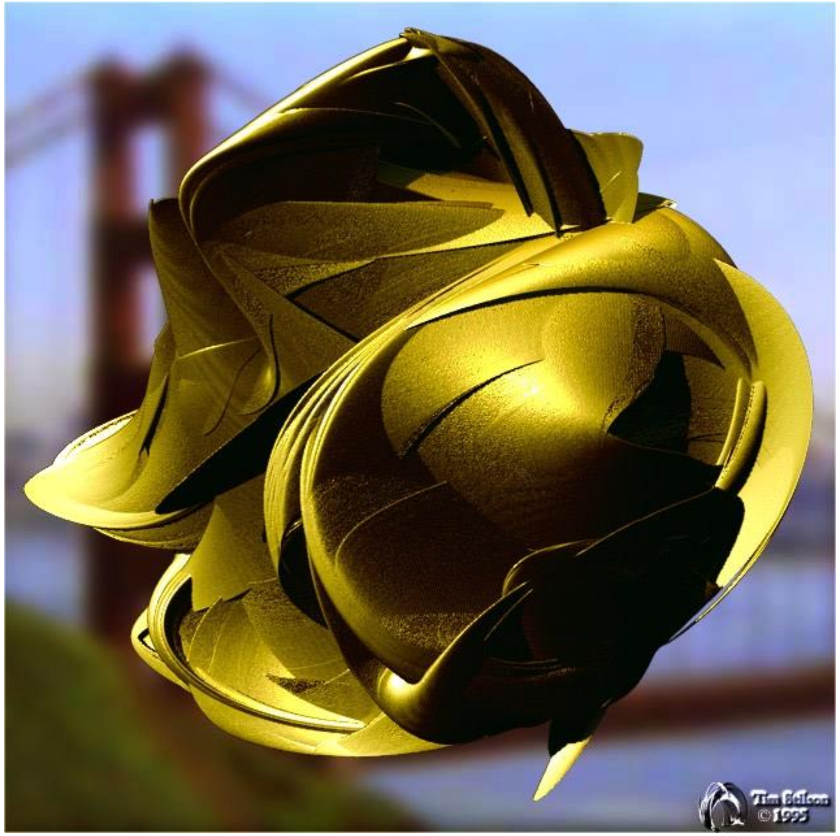


Varying starting position

Other SA

Other SA 2



End

End of Part 9